Chapter 11

Dynamic Behavior and Stability of Closed-Loop Control Systems

- In this chapter we consider the dynamic behavior of processes that are operated using feedback control.
- This combination of the process, the feedback controller, and the instrumentation is referred to as a *feedback control loop* or a *closed-loop system.*

Block Diagram Representation

To illustrate the development of a block diagram, we return to a previous example, the stirred-tank blending process considered in earlier chapters.

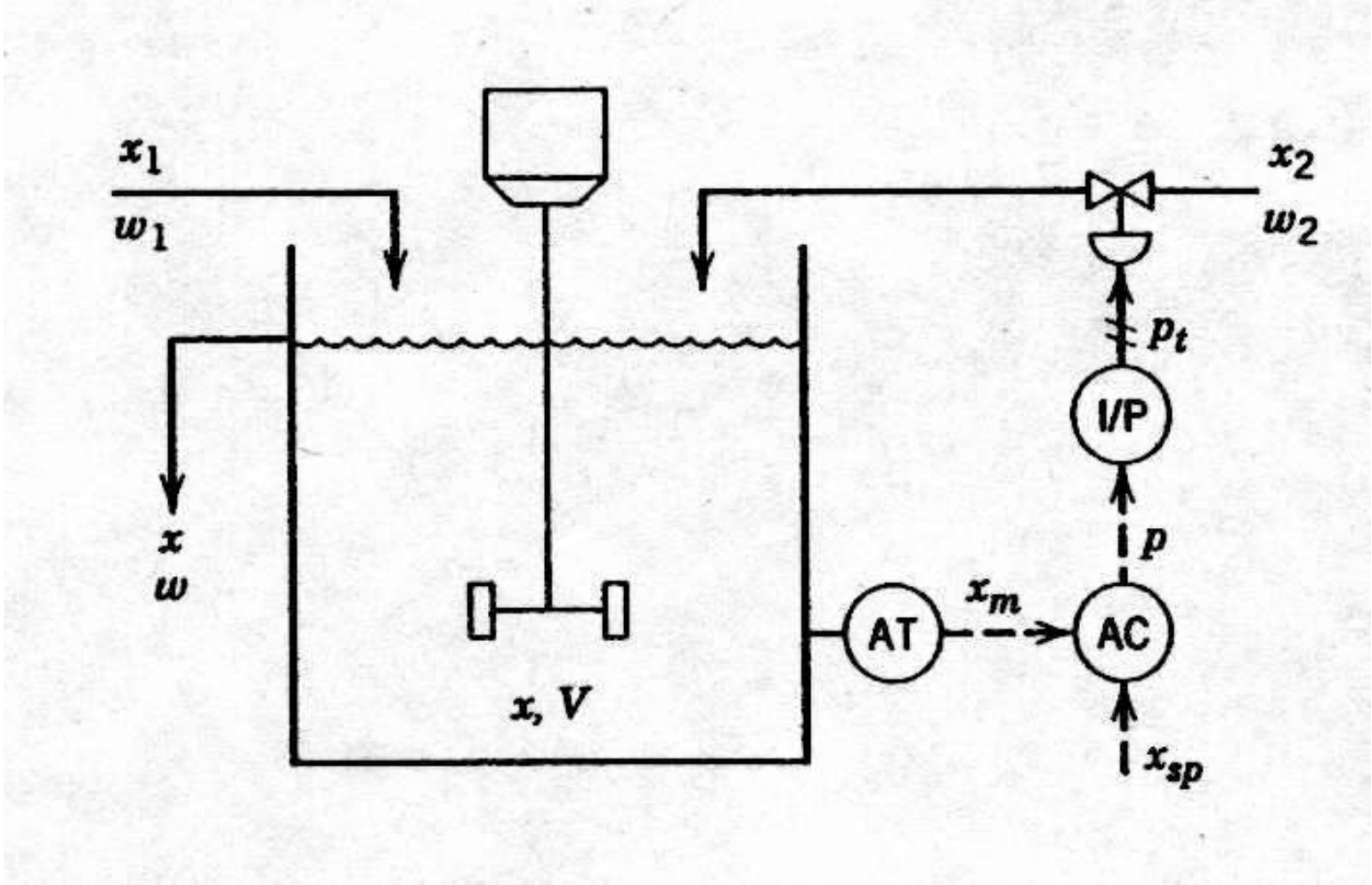

Figure 11.1 Composition control system for a stirred-tank blending process.

Next, we develop a transfer function for each of the five elements in the feedback control loop. For the sake of simplicity, flow rate w_1 is assumed to be constant, and the system is initially operating at the nominal steady rate.

Process

In section 4.3 the approximate dynamic model of a stirred-tank blending system was developed:

$$X'(s) = \left(\frac{K_1}{\tau s + 1}\right) X_1'(s) + \left(\frac{K_2}{\tau s + 1}\right) W_2'(s) \qquad (11\text{-}1)$$

where

$$\tau = \frac{V\rho}{\bar{w}}, \qquad K_1 = \frac{\bar{w}_1}{\bar{w}}, \quad \text{and} \quad K_2 = \frac{1 - \bar{x}}{\bar{w}} \qquad (11\text{-}2)$$

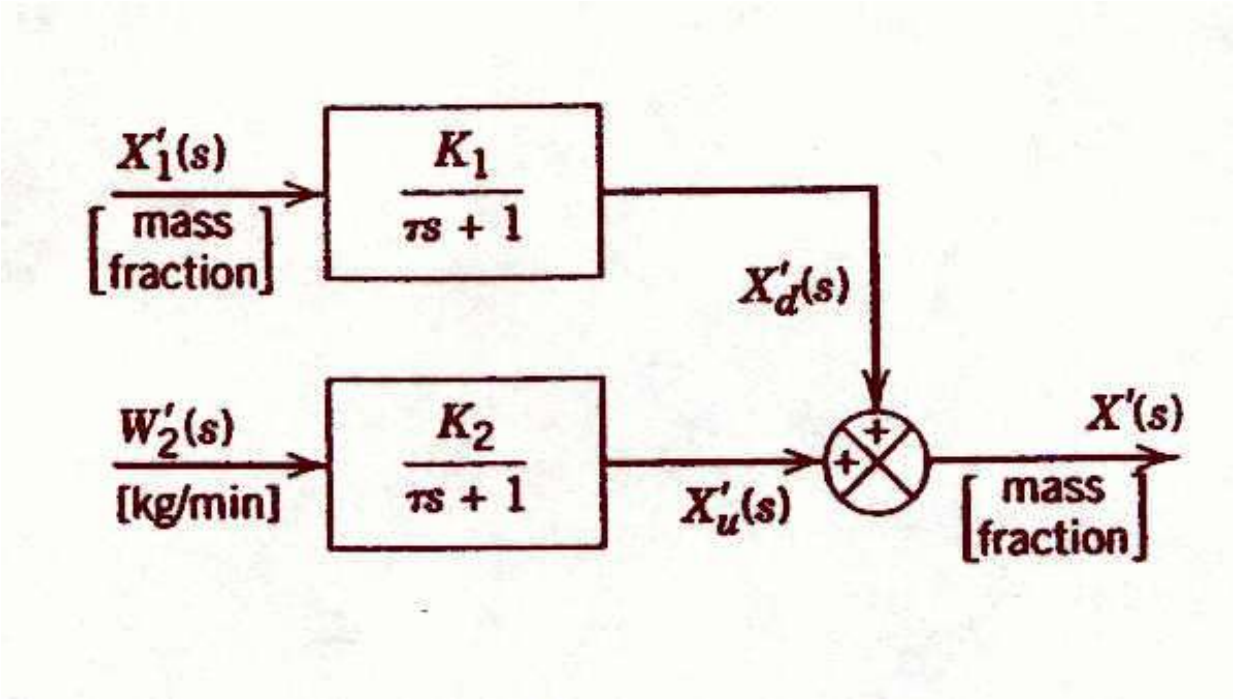

Figure 11.2 Block diagram of the process.

Composition Sensor-Transmitter (Analyzer)

We assume that the dynamic behavior of the composition sensor-transmitter can be approximated by a first-order transfer function:

$$\frac{X'_m(s)}{X'(s)} = \frac{K_m}{\tau_m s + 1} \qquad (11\text{-}3)$$

Controller

Suppose that an electronic proportional plus integral controller is used. From Chapter 8, the controller transfer function is

$$\frac{P'(s)}{E(s)} = K_c\left(1 + \frac{1}{\tau_I s}\right) \qquad (11\text{-}4)$$

where $P'(s)$ and $E(s)$ are the Laplace transforms of the controller output $p'(t)$ and the error signal $e(t)$. Note that p' and e are electrical signals that have units of mA, while K_c is dimensionless. The error signal is expressed as

$$e(t) = \tilde{x}'_{sp}(t) - x'_m(t) \qquad (11\text{-}5)$$

or after taking Laplace transforms,

$$E(s) = \tilde{X}'_{sp}(s) - X'_m(s) \qquad (11\text{-}6)$$

The symbol $\tilde{x}'_{sp}(t)$ denotes the *internal set-point* composition expressed as an equivalent electrical current signal. This signal is used internally by the controller. $\tilde{x}'_{sp}(t)$ is related to the actual composition set point $x'_{sp}(t)$ by the composition sensor-transmitter gain K_m:

$$\tilde{x}'_{sp}(t) = K_m x'_{sp}(t) \qquad (11\text{-}7)$$

Thus

$$\frac{\tilde{X}'_{sp}(s)}{X'_{sp}(s)} = K_m \qquad (11\text{-}8)$$

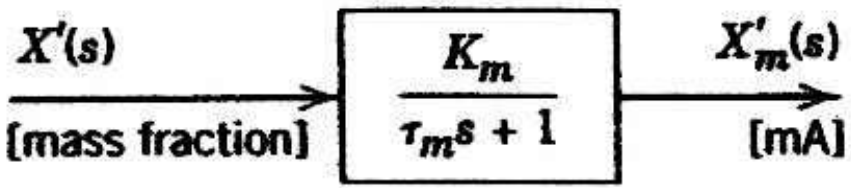

Figure 11.3 Block diagram for the composition sensor-transmitter (analyzer).

Current-to-Pressure (I/P) Transducer

Because transducers are usually designed to have linear characteristics and negligible (fast) dynamics, we assume that the transducer transfer function merely consists of a steady-state gain K_{IP}:

$$\frac{P_t'(s)}{P'(s)} = K_{IP} \qquad (11\text{-}9)$$

Control Valve

As discussed in Section 9.2, control valves are usually designed so that the flow rate through the valve is a nearly linear function of the signal to the valve actuator. Therefore, a first-order transfer function usually provides an adequate model for operation of an installed valve in the vicinity of a nominal steady state. Thus, we assume that the control valve can be modeled as

$$\frac{W_2'(s)}{P_t'(s)} = \frac{K_v}{\tau_v s + 1} \qquad (11\text{-}10)$$

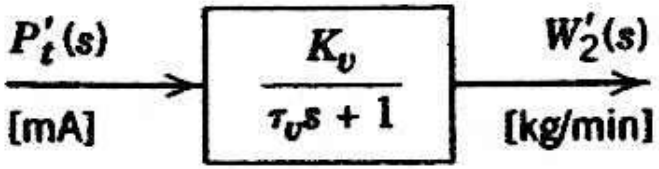

Figure 11.5 Block diagram for the I/P transducer.

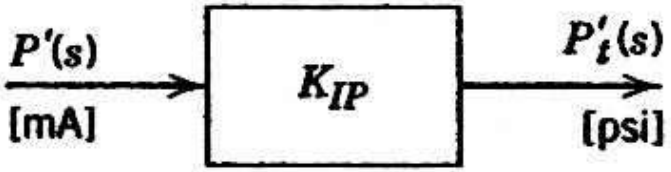

Figure 11.6 Block diagram for the control valve.

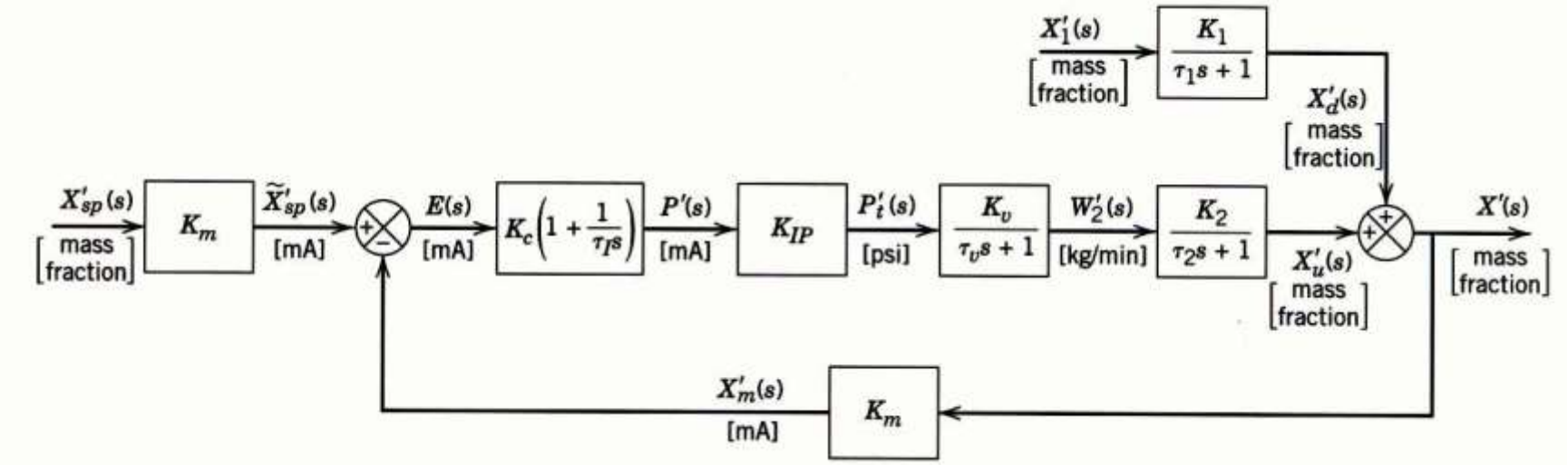

Figure 11.7 Block diagram for the entire blending process composition control system.

Closed-Loop Transfer Functions

The block diagrams considered so far have been specifically developed for the stirred-tank blending system. The more general block diagram in Fig. 11.8 contains the standard notation:

Y = controlled variable

U = manipulated variable

D = disturbance variable (also referred to as *load variable)*

P = controller output

E = error signal

Y_m = measured value of Y

Y_{sp} = set point

$\tilde{Y}_{sp}$ = internal set point (used by the controller)

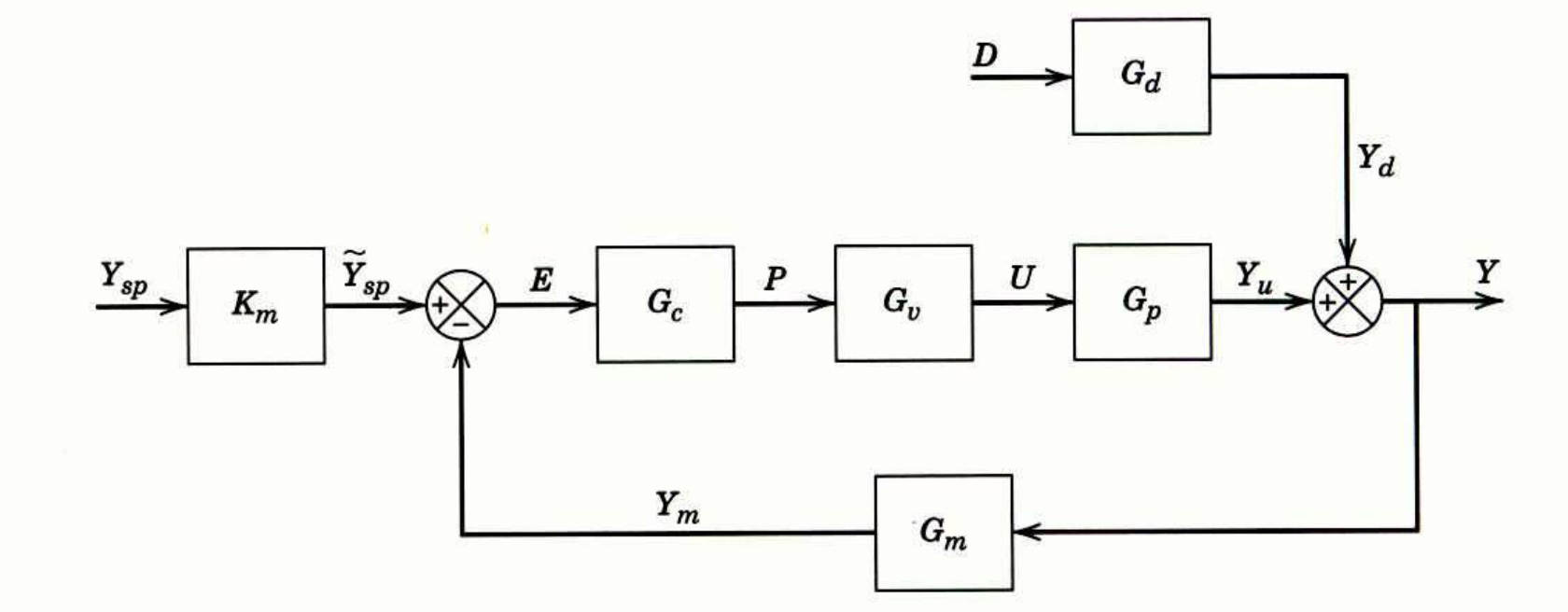

Figure 11.8 Standard block diagram of a feedback control system.

Y_u = change in Y due to U

Y_d = change in Y due to D

G_c = controller transfer function

G_v = transfer function for final control element (including K_{IP}, if required)

G_p = process transfer function

G_d = disturbance transfer function

G_m = transfer function for measuring element and transmitter

K_m = steady-state gain for G_m

Block Diagram Reduction

In deriving closed-loop transfer functions, it is often convenient to combine several blocks into a single block. For example, consider the three blocks in series in Fig. 11.10. The block diagram indicates the following relations:

$$X_1 = G_1 U$$
$$X_2 = G_2 X_1 \qquad (11\text{-}11)$$
$$X_3 = G_3 X_2$$

By successive substitution,

$$X_3 = G_3 G_2 G_1 U \qquad (11\text{-}12)$$

or

$$X_3 = GU \qquad (11\text{-}13)$$

where $G \square G_3 G_2 G_1$.

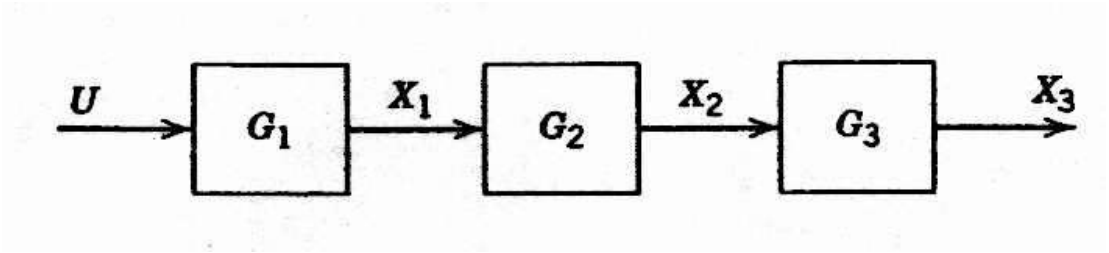

Figure 11.10 Three blocks in series.

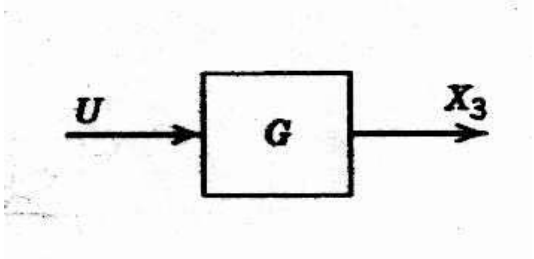

Figure 11.11 Equivalent block diagram.

Set-Point Changes

Next we derive the closed-loop transfer function for set-point changes. The closed-loop system behavior for set-point changes is also referred to as the servomechanism (*servo*) *problem* in the control literature.

$$Y = Y_d + Y_u \tag{11-14}$$

$$Y_d = G_d D = 0 \text{ (because } D = 0) \tag{11-15}$$

$$Y_u = G_p U \tag{11-16}$$

Combining gives

$$Y = G_p U \tag{11-17}$$

Figure 11.8 also indicates the following input/output relations for the individual blocks:

$$U = G_v P \tag{11-18}$$

$$P = G_c E \tag{11-19}$$

$$E = \tilde{Y}_{sp} - Y_m \tag{11-20}$$

$$\tilde{Y}_{sp} = K_m Y_{sp} \tag{11-21}$$

$$Y_m = G_m Y \tag{11-22}$$

Combining the above equations gives

$$Y = G_p G_v P = G_p G_v G_c E \tag{11-23}$$

$$= G_p G_v G_c \left(\tilde{Y}_{sp} - Y_m \right) \tag{11-24}$$

$$= G_p G_v G_c \left(K_m Y_{sp} - G_m Y \right) \tag{11-25}$$

Rearranging gives the desired closed-loop transfer function,

$$\frac{Y}{Y_{sp}} = \frac{K_m G_c G_v G_p}{1 + G_c G_v G_p G_m} \qquad (11\text{-}26)$$

Disturbance Changes

Now consider the case of disturbance changes, which is also referred to as the *regulator problem* since the process is to be *regulated* at a constant set point. From Fig. 11.8,

$$Y = Y_d + Y_u = G_d D + G_p U \qquad (11\text{-}27)$$

Substituting (11-18) through (11-22) gives

$$Y = G_d D + G_p U = G_p G_v G_c \left(K_m Y_{sp} - G_m Y \right) \qquad (11\text{-}28)$$

Because $Y_{sp} = 0$ we can arrange (11-28) to give the closed-loop transfer function for disturbance changes:

$$\frac{Y}{D} = \frac{G_d}{1 + G_c G_v G_p G_m} \qquad (11\text{-}29)$$

A comparison of Eqs. 11-26 and 11-29 indicates that both closed-loop transfer functions have the same denominator, $1 + G_c G_v G_p G_m$. The denominator is often written as $1 + G_{OL}$ where G_{OL} is the *open-loop transfer function*, $G_{OL} \equiv G_c G_v G_p G_m$.

At different points in the above derivations, we assumed that $D = 0$ or $Y_{sp} = 0$, that is, that one of the two inputs was constant.

But suppose that $D \neq 0$ and $Y_{sp} \neq 0$, as would be the case if a disturbance occurs during a set-point change. To analyze this situation, we rearrange Eq. 11-28 and substitute the definition of G_{OL} to obtain

$$Y = \frac{G_d}{1 + G_{OL}} D + \frac{K_m G_c G_v G_p}{1 + G_{OL}} Y_{sp} \qquad (11\text{-}30)$$

Thus, the response to simultaneous disturbance variable and set-point changes is merely the sum of the individual responses, as can be seen by comparing Eqs. 11-26, 11-29, and 11-30.

This result is a consequence of the Superposition Principle for linear systems.

General Expression for Feedback Control Systems

Closed-loop transfer functions for more complicated block diagrams can be written in the general form:

$$\frac{Z}{Z_i} = \frac{\Pi_f}{1 + \Pi_e} \qquad (11\text{-}31)$$

where:

Z is the output variable or any internal variable within the control loop

Z_i is an input variable (e.g., Y_{sp} or D)

Π_f = product of the transfer functions in the *forward* path from Z_i to Z

Π_e = product of *every* transfer function in the feedback loop

Example 11.1

Find the closed-loop transfer function Y/Y_{sp} for the complex control system in Figure 11.12. Notice that this block diagram has two feedback loops and two disturbance variables. This configuration arises when the cascade control scheme of Chapter 16 is employed.

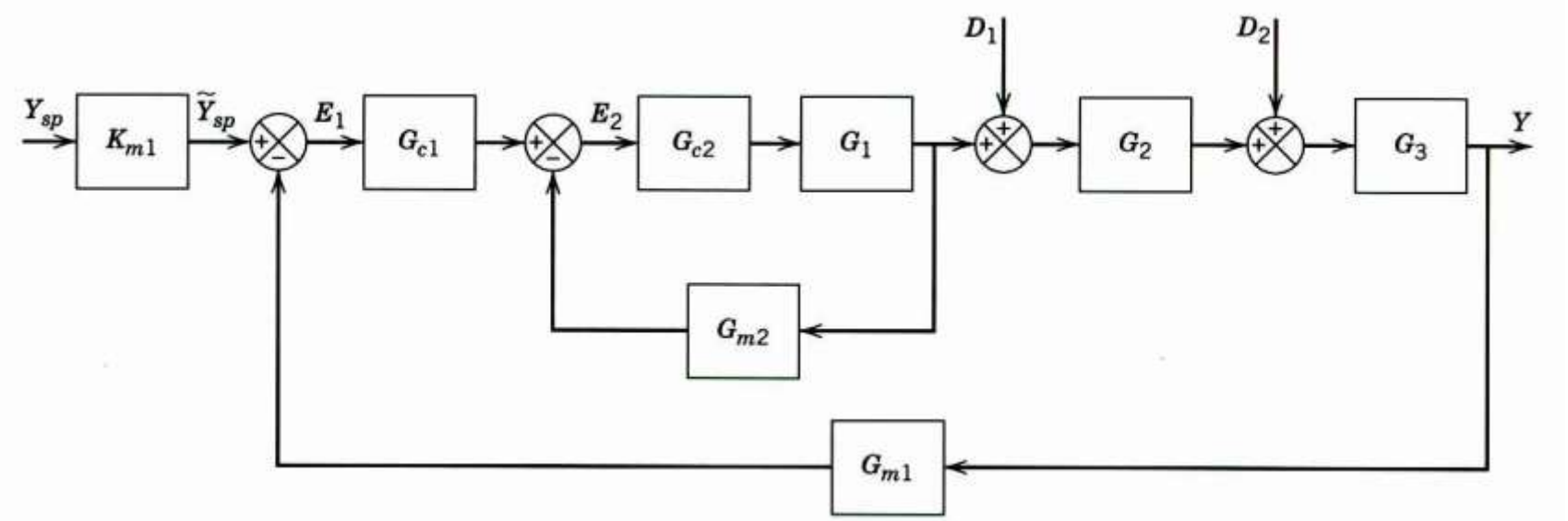

Figure 11.12 Complex control system.

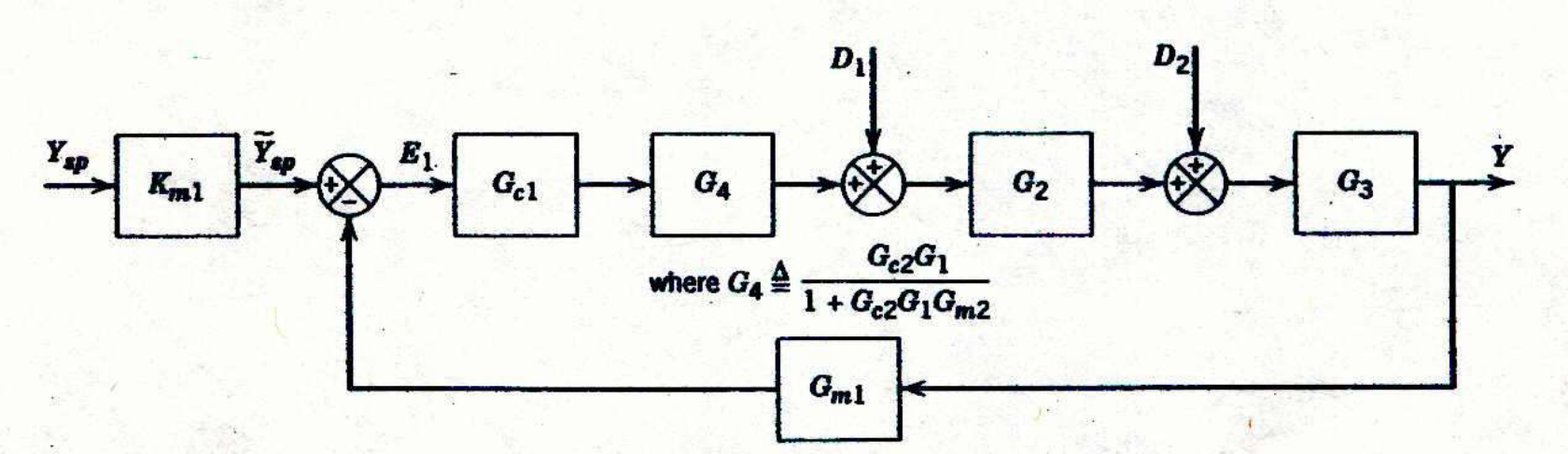

Figure 11.13 Block diagram for reduced system.

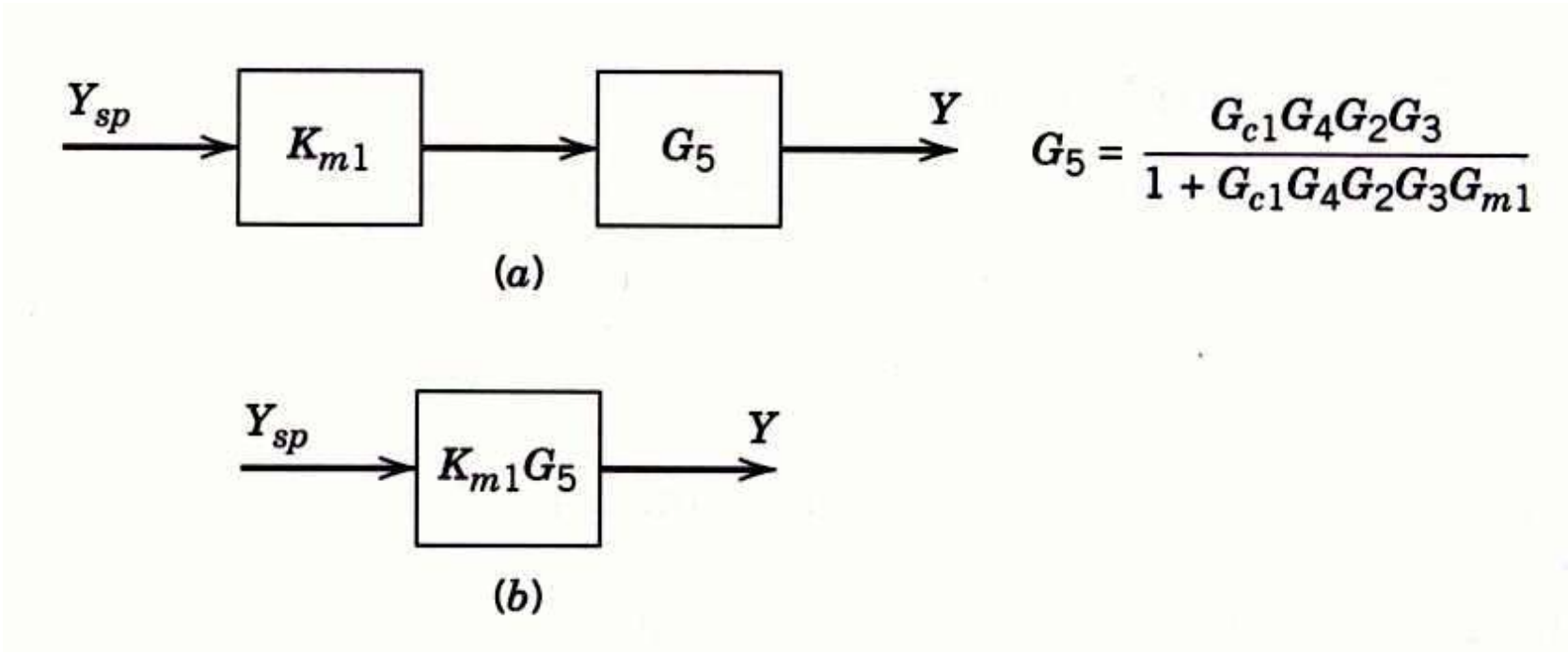

Figure 11.14 Final block diagrams for Example 11.1.

Solution

Using the general rule in (11-31), we first reduce the inner loop to a single block as shown in Fig. 11.13. To solve the servo problem, set $D_1 = D_2 = 0$. Because Fig. 11.13 contains a single feedback loop, use (11-31) to obtain Fig. 11.14a. The final block diagram is shown in Fig. 11.14b with $Y/Y_{sp} = K_{m1}G_5$. Substitution for G_4 and G_5 gives the desired closed-loop transfer function:

$$\frac{Y}{Y_{sp}} = \frac{K_{m1}G_{c1}G_{c2}G_1G_2G_3}{1+G_{c2}G_1G_{m2}+G_{c1}G_2G_3G_{m1}G_{c2}G_1}$$

Closed-Loop Responses of Simple Control Systems

In this section we consider the dynamic behavior of several elementary control problems for disturbance variable and set-point changes.

The transient responses can be determined in a straightforward manner if the closed-loop transfer functions are available.

Consider the liquid-level control system shown in Fig. 11.15. The liquid level is measured and the level transmitter (LT) output is sent to a feedback controller (LC) that controls liquid level by adjusting volumetric flow rate q_2. A second inlet flow rate q_1 is the disturbance variable. Assume:

1. The liquid density ρ and the cross-sectional area of the tank A are constant.
2. The flow-head relation is linear, $q_3 = h/R$.
3. The level transmitter, I/P transducer, and control valve have negligible dynamics.
4. An electronic controller with input and output in % is used (full scale = 100%).

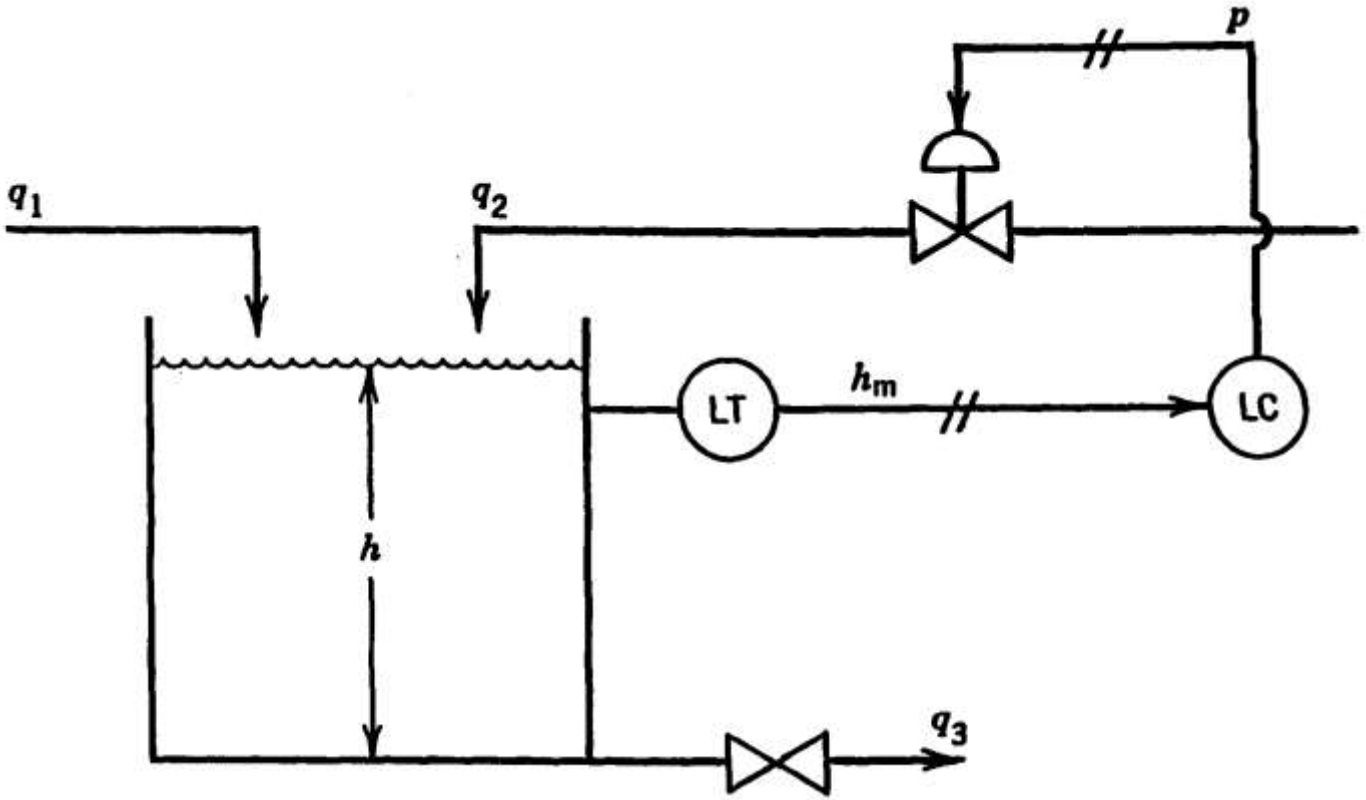

Figure 11.15 Liquid-level control system.

Derivation of the process and disturbance transfer functions directly follows Example 4.4. Consider the unsteady-state mass balance for the tank contents:

$$\rho A\frac{dh}{dt} = \rho q_1 + \rho q_2 - \rho q_3 \qquad (11\text{-}32)$$

Substituting the flow-head relation, $q_3 = h/R$, and introducing deviation variables gives

$$A\frac{dh'}{dt} = q_1' + q_2' - \frac{h'}{R} \qquad (11\text{-}33)$$

Thus, we obtain the transfer functions

$$\frac{H'(s)}{Q_2'(s)} = G_p(s) = \frac{K_p}{\tau s + 1} \qquad (11\text{-}34)$$

$$\frac{H'(s)}{Q_1'(s)} = G_d(s) = \frac{K_p}{\tau s + 1} \qquad (11\text{-}35)$$

where $K_p = R$ and $\tau = RA$. Note that $G_p(s)$ and $G_d(s)$ are identical because q_1 and q_2 are both inlet flow rates and thus have the same effect on h.

Proportional Control and Set-Point Changes

If a proportional controller is used, then $G_c(s) = K_c$. From Fig. 11.6 and the material in the previous section, it follows that the closed-loop transfer function for set-point changes is given by

$$\frac{H'(s)}{H_{sp}'(s)} = \frac{K_c K_v K_p K_m / (\tau s + 1)}{1 + K_c K_v K_p K_m / (\tau s + 1)} \qquad (11\text{-}36)$$

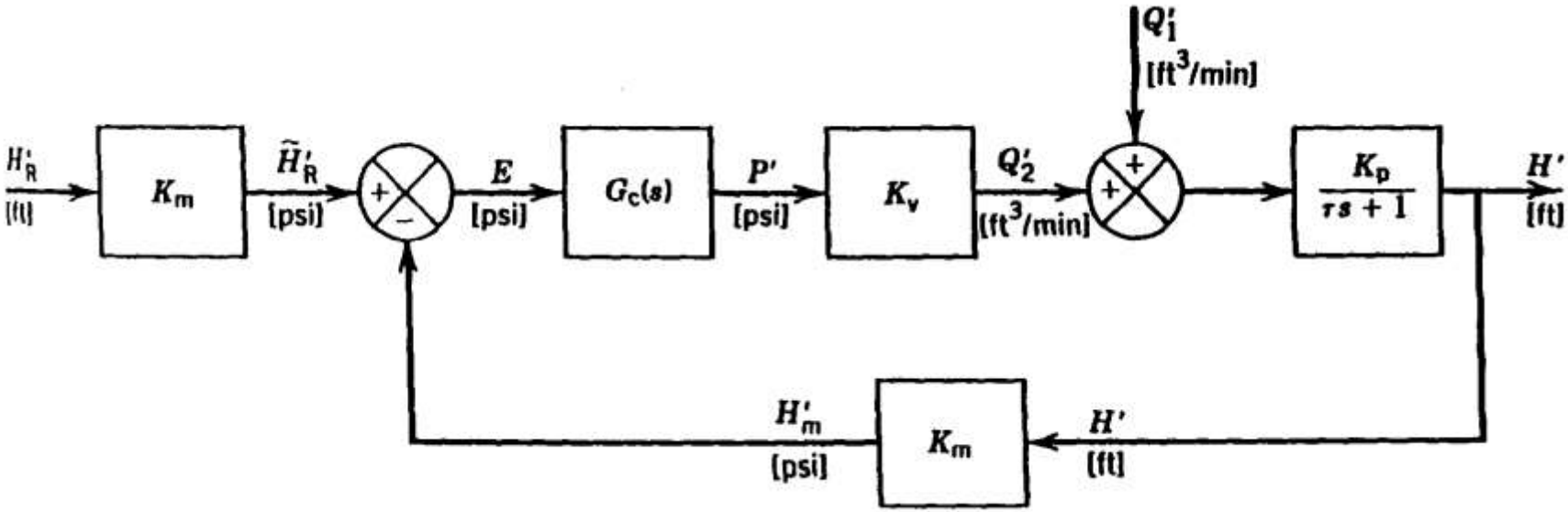

Figure 11.16 Block diagram for level control system.

This relation can be rearranged in the standard form for a first-order transfer function,

$$\frac{H'(s)}{H'_{sp}(s)} = \frac{K_1}{\tau_1 s + 1} \tag{11-37}$$

where:

$$K_1 = \frac{K_{OL}}{1 + K_{OL}} \tag{11-38}$$

$$\tau_1 = \frac{\tau}{1 + K_{OL}} \tag{11-39}$$

and the *open-loop gain* K_{OL} is given by

$$K_{OL} = K_c K_v K_p K_m \tag{11-40}$$

From Eq. 11-37 it follows that the closed-loop response to a unit step change of magnitude M in set point is given by

$$h'(t) = K_1 M\left(1 - e^{-t/\tau_1}\right) \qquad (11\text{-}41)$$

This response is shown in Fig. 11.17. Note that a steady-state error or *offset* exists because the new steady-state value is $K_1 M$ rather than the desired value of M. The offset is defined as

$$\text{offset} \equiv h'_{sp}(\infty) - h'(\infty) \qquad (11\text{-}42)$$

For a step change of magnitude M in set point, $h'_{sp}(\infty) = M$. From (11-41), it is clear that $h'(\infty) = K_1 M$. Substituting these values and (11-38) into (11-42) gives

$$\text{offset} = M - K_1 M = \frac{M}{1 + K_{OL}} \qquad (11\text{-}43)$$

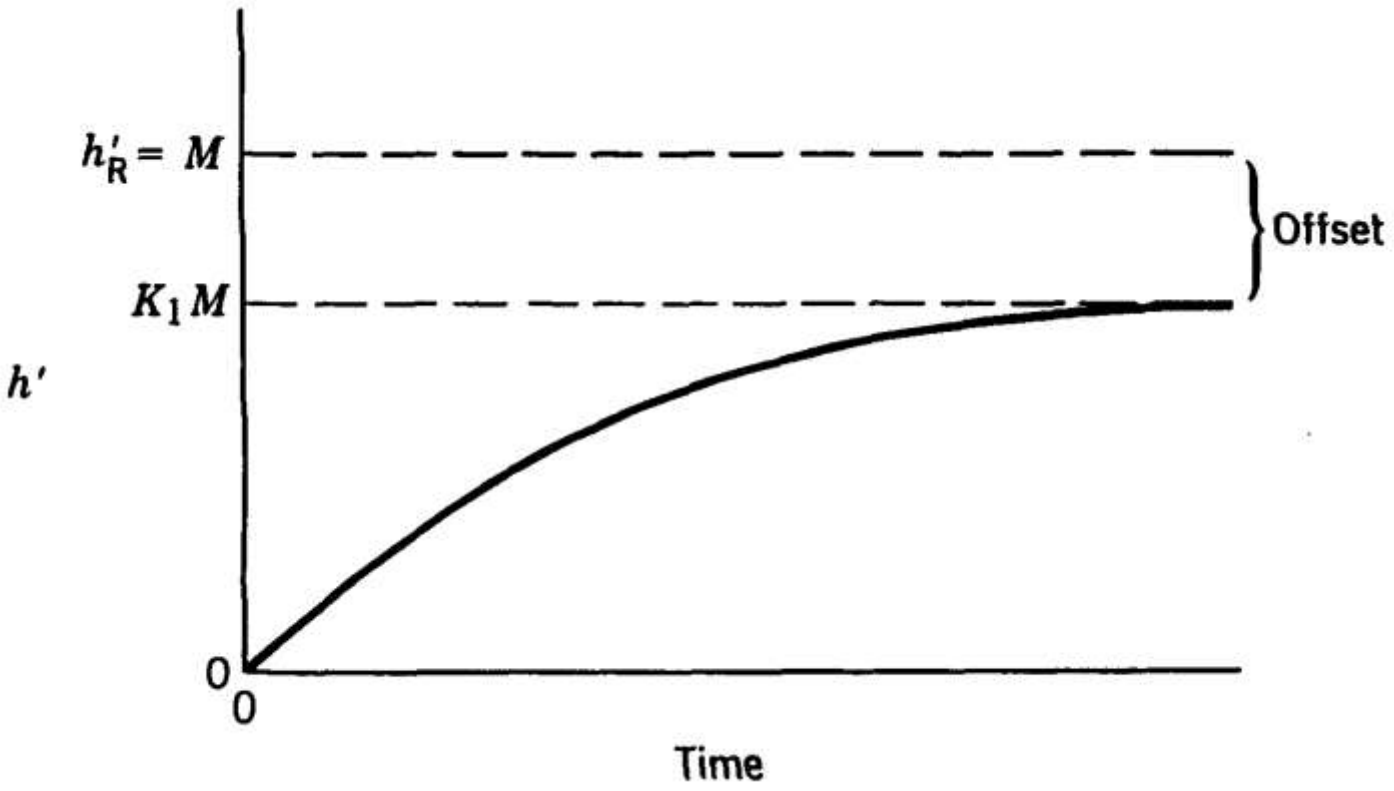

Figure 11.17 Step response for proportional control (set-point change).

Proportional Control and Disturbance Changes

From Fig. 11.16 and Eq. 11-29 the closed-loop transfer function for disturbance changes with proportional control is

$$\frac{H'(s)}{Q_1'(s)} = \frac{K_p/(\tau s+1)}{1+K_{OL}/(\tau s+1)} \qquad (11\text{-}53)$$

Rearranging gives

$$\frac{H'(s)}{Q_1'(s)} = \frac{K_2}{\tau_1 s+1} \qquad (11\text{-}54)$$

where τ_1 is defined in (11-39) and K_2 is given by

$$K_2 = \frac{K_p}{1+K_{OL}} \qquad (11\text{-}55)$$

- A comparison of (11-54) and (11-37) indicates that both closed-loop transfer functions are first-order and have the same time constant.
- However, the steady-state gains, K_1 and K_2, are different.
- From Eq. 11-54 it follows that the closed-loop response to a step change in disturbance of magnitude M is given by

$$h'(t) = K_2 M \left(1 - e^{-t/\tau_1}\right) \qquad (11\text{-}56)$$

The offset can be determined from Eq. 11-56. Now $h'_{sp}(\infty) = 0$ since we are considering disturbance changes and $h'(\infty) = K_2 M$ for a step change of magnitude M.

Thus,

$$\text{offset} = 0 - h'(\infty) = -K_2 M = -\frac{K_p M}{1 + K_{OL}} \qquad (11\text{-}57)$$

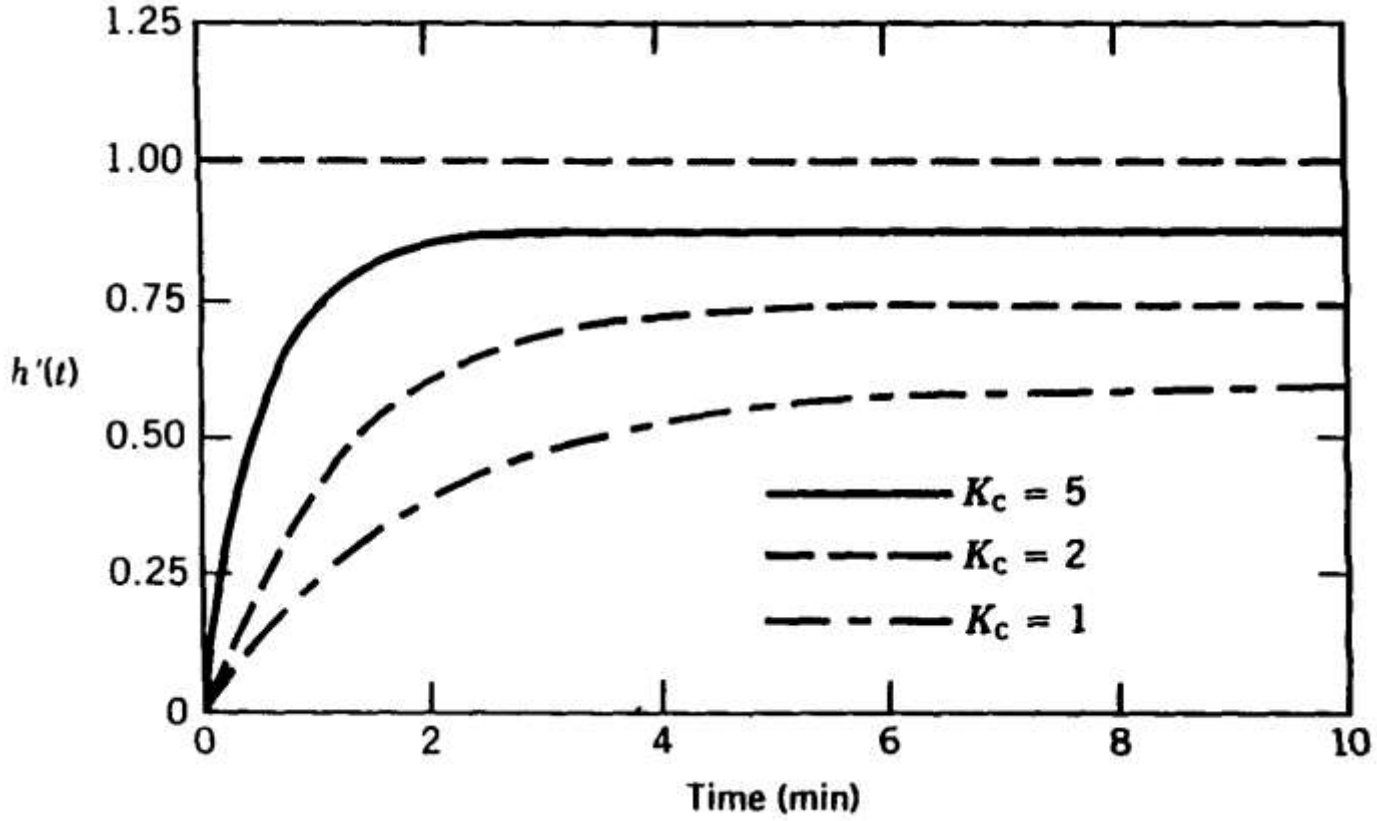

Figure 11.18 Set-point responses for Example 11.2.

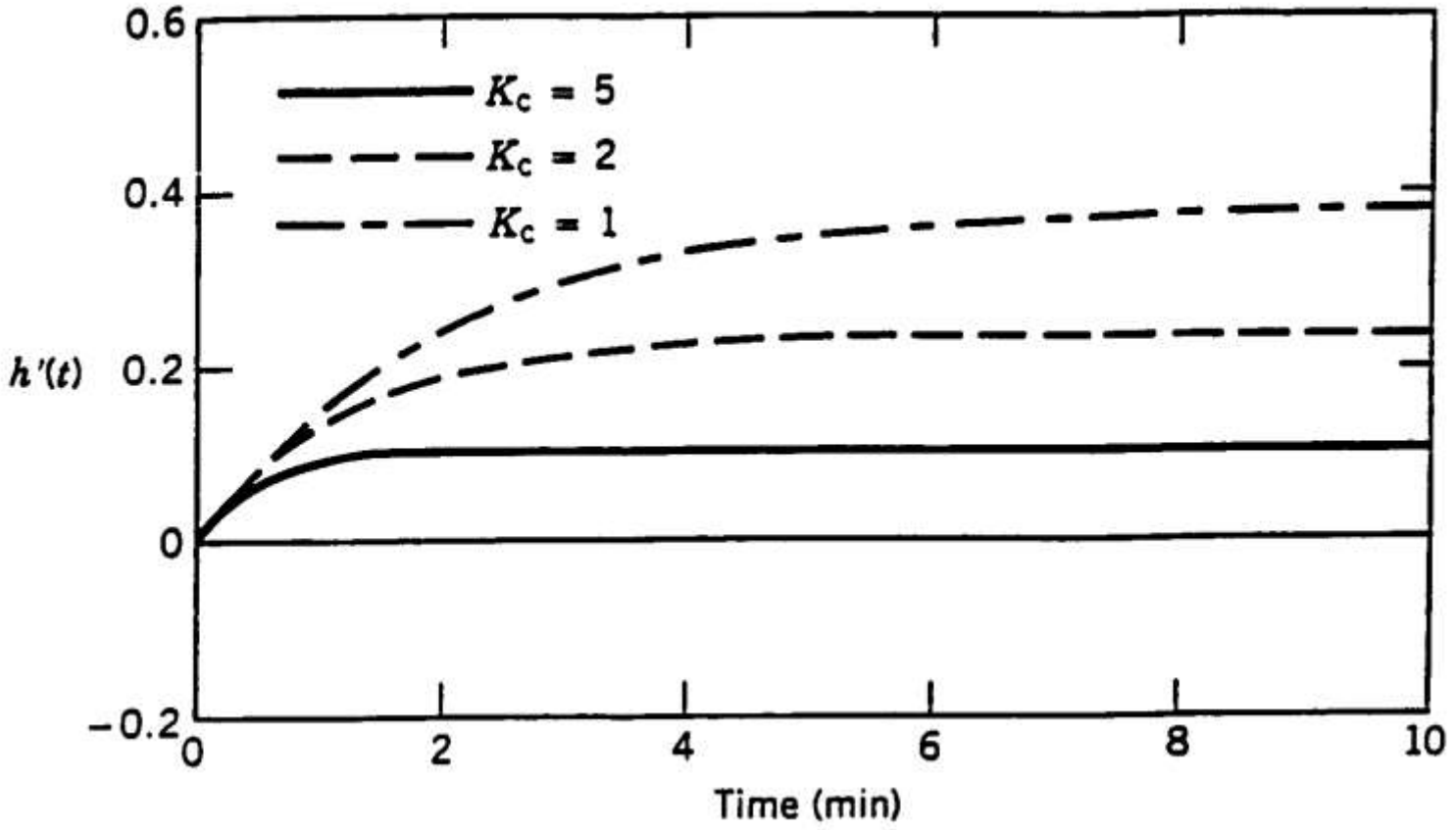

Figure 11.19 Load responses for Example 11.3.

PI Control and Disturbance Changes

For PI control, $G_c(s) = K_c(1+1/\tau_I s)$. The closed-loop transfer function for disturbance changes can then be derived from Fig. 11.16:

$$\frac{H'(s)}{Q_1'(s)} = \frac{K_p/(\tau s+1)}{1+K_{OL}(1+1/\tau_I s)/(\tau s+1)} \qquad (11\text{-}58)$$

Clearing terms in the denominator gives

$$\frac{H'(s)}{Q_1'(s)} = \frac{K_{p\tau_I s}}{\tau_I s(\tau s+1)+K_{OL}\tau_I s} \qquad (11\text{-}59)$$

Further rearrangement allows the denominator to be placed in the standard form for a second-order transfer function:

$$\frac{H'(s)}{Q_1'(s)} = \frac{K_3 s}{\tau_3^2 s^2 + 2\zeta_3\tau_3 s + 1} \qquad (11\text{-}60)$$

where

$$K_3 = \tau_I / K_c K_v K_m \qquad (11\text{-}61)$$

$$\zeta_3 = \frac{1}{2}\left(\frac{1+K_{OL}}{\sqrt{K_{OL}}}\right)\sqrt{\frac{\tau_I}{\tau}} \qquad (11\text{-}62)$$

$$\tau_3 = \sqrt{\tau\tau_I / K_{OL}} \qquad (11\text{-}63)$$

For a unit step change in disturbance, $Q_1'(s) = 1/s$, and (11-59) becomes

$$H'(s) = \frac{K_3}{\tau_3^2 s^2 + 2\zeta_3\tau_3 s + 1} \qquad (11\text{-}64)$$

For $0 < \zeta_3 < 1$, the response is a damped oscillation that can be described by

$$h'(t) = \frac{K_3}{\tau_3\sqrt{1-\zeta_3^2}} e^{-\zeta_3 t/\tau_3} \sin\left[\sqrt{1-\zeta_3^2}\, t/\tau_3\right] \qquad (11\text{-}65)$$

PI Control of an Integrating Process

Consider the liquid-level control system shown in Fig. 11.22. This system differs from the previous example in two ways:

1. the exit line contains a pump and
2. the manipulated variable is the exit flow rate rather than an inlet flow rate.

In Section 5.3 we saw that a tank with a pump in the exit stream can act as an integrator with respect to flow rate changes because

$$\frac{H'(s)}{Q_3'(s)} = G_p(s) = -\frac{1}{As} \qquad (11\text{-}66)$$

$$\frac{H'(s)}{Q_1'(s)} = G_d(s) = \frac{1}{As} \qquad (11\text{-}67)$$

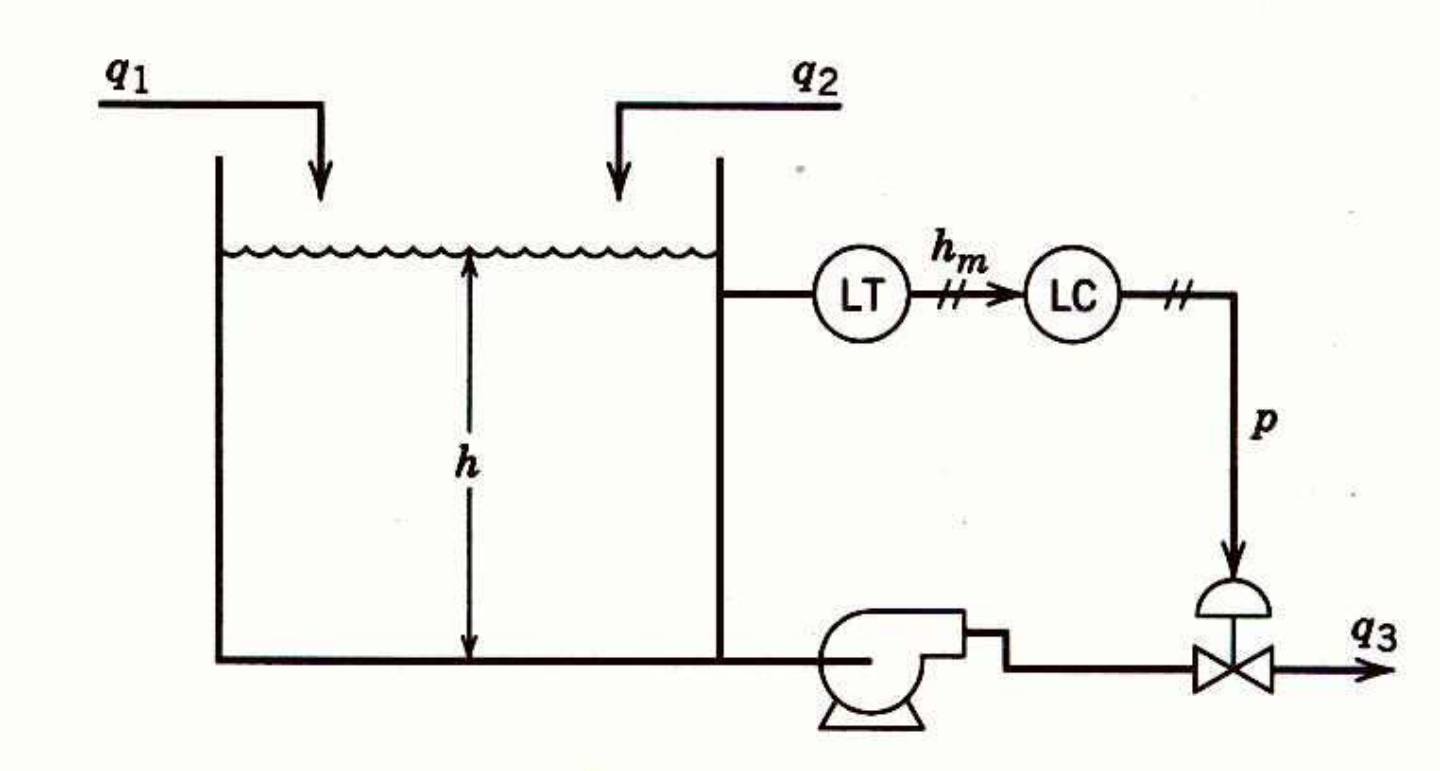

Figure 11.22 Liquid-level control system with pump in exit line.

If the level transmitter and control valve in Eq. 11.22 have negligible dynamics, the $G_m(s) = K_m$ and $G_v(s) = K_v$. For PI control, $G_c(s) = K_c(1+1/\tau_I s)$. Substituting these expressions into the closed-loop transfer function for disturbance changes

$$\frac{H'(s)}{Q_1'(s)} = \frac{G_d}{1+G_c G_v G_p G_m} \qquad (11\text{-}68)$$

and rearranging gives

$$\frac{H'(s)}{Q_1'(s)} = \frac{K_4 s}{\tau_4^2 s^2 + 2\zeta_4 \tau_4 s + 1} \qquad (11\text{-}69)$$

where

$$K_4 = -\tau / K_c K_v K_m \qquad (11\text{-}70)$$

$$\tau_4 = \sqrt{\tau_I / K_{OL}} \qquad (11\text{-}71)$$

$$\zeta_4 = 0.5\sqrt{K_{OL}\tau_I} \qquad (11\text{-}72)$$

And $K_{OL} = K_c K_v K_p K_m$ with K_p = - 1/A.

Stability of Closed-Loop Control Systems

Example 11.4

Consider the feedback control system shown in Fig. 11.8 with the following transfer functions:

$$G_c = K_c \qquad G_v = \frac{1}{2s+1} \tag{11-73}$$

$$G_p = G_d = \frac{1}{5s+1} \qquad G_m = \frac{1}{s+1} \tag{11-74}$$

Show that the closed-loop system produces unstable responses if controller gain K_c is too large.

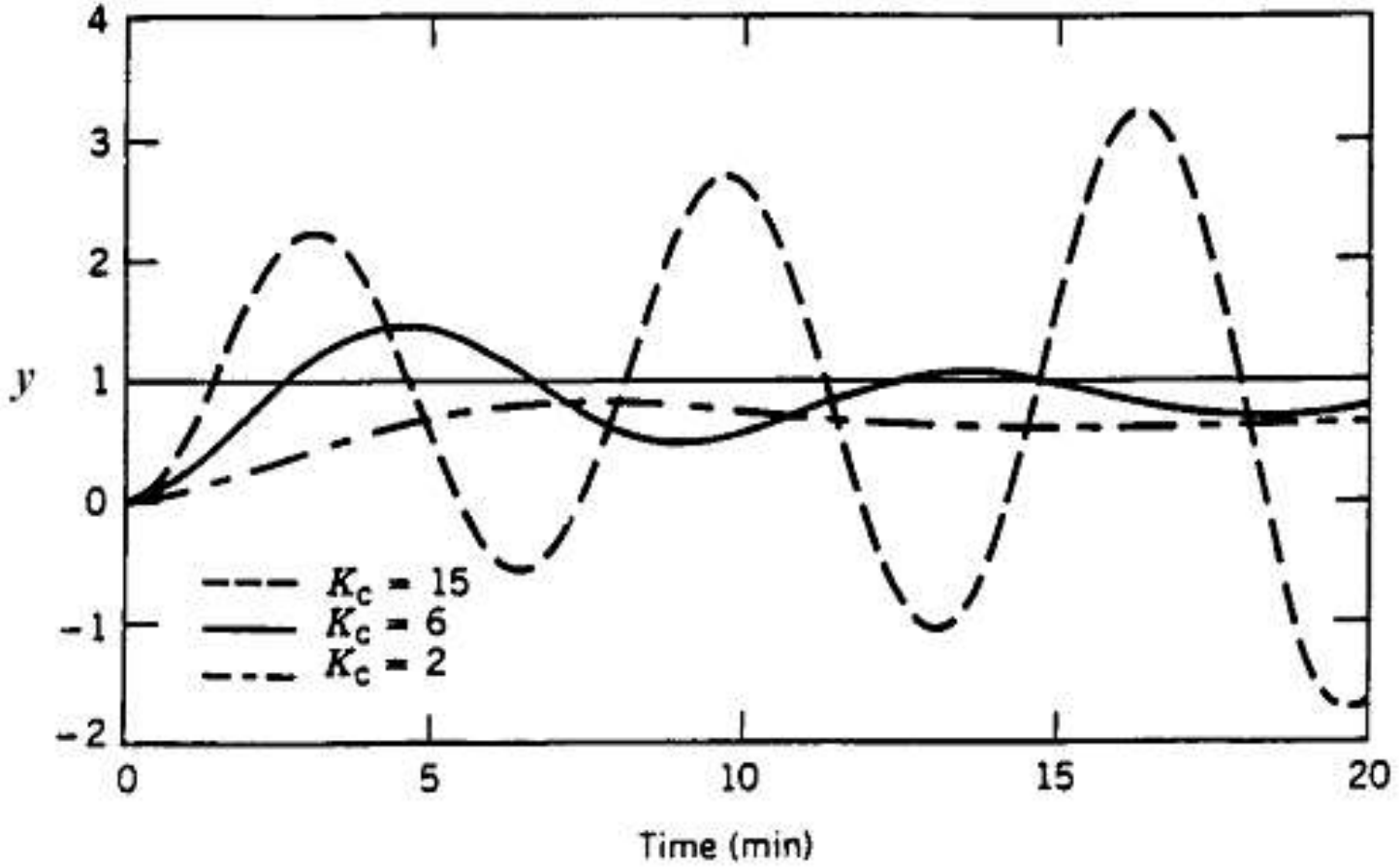

Figure 11.23. Effect of controller gains on closed-loop response to a unit step change in set point (example 11.1).

Stability

- Most industrial processes are stable without feedback control. Thus, they are said to be *open-loop stable* or *self-regulating*.
- An open-loop stable process will return to the original steady state after a transient disturbance (one that is not sustained) occurs.
- By contrast there are a few processes, such as exothermic chemical reactors, that can be *open-loop unstable*.

Definition of Stability. *An unconstrained linear system is said to be stable if the output response is bounded for all bounded inputs. Otherwise it is said to be unstable.*

Characteristic Equation

As a starting point for the stability analysis, consider the block diagram in Fig. 11.8. Using block diagram algebra that was developed earlier in this chapter, we obtain

$$Y = \frac{K_m G_c G_v G_p}{1 + G_{OL}} Y_{sp} + \frac{G_d}{1 + G_{OL}} D \qquad (11\text{-}80)$$

where G_{OL} is the open-loop transfer function, $G_{OL} = G_c G_v G_p G_m$. For the moment consider set-point changes only, in which case Eq. 11-80 reduces to the closed-loop transfer function,

$$\frac{Y}{Y_{sp}} = \frac{K_m G_c G_v G_p}{1 + G_{OL}} \qquad (11\text{-}81)$$

Comparing Eqs. 11-81 and 11-82 indicates that the poles are also the roots of the following equation, which is referred to as the *characteristic equation* of the closed-loop system:

$$1 + G_{OL} = 0 \qquad (11\text{-}83)$$

General Stability Criterion. *The feedback control system in Fig. 11.8 is stable if and only if* all *roots of the characteristic equation are negative or have negative real parts. Otherwise, the system is unstable.*

Example 11.8

Consider a process, $G_p = 0.2/\text{-}s + 1)$, and thus is open-loop unstable. If $G_v = G_m = 1$, determine whether a proportional controller can stabilize the closed-loop system.

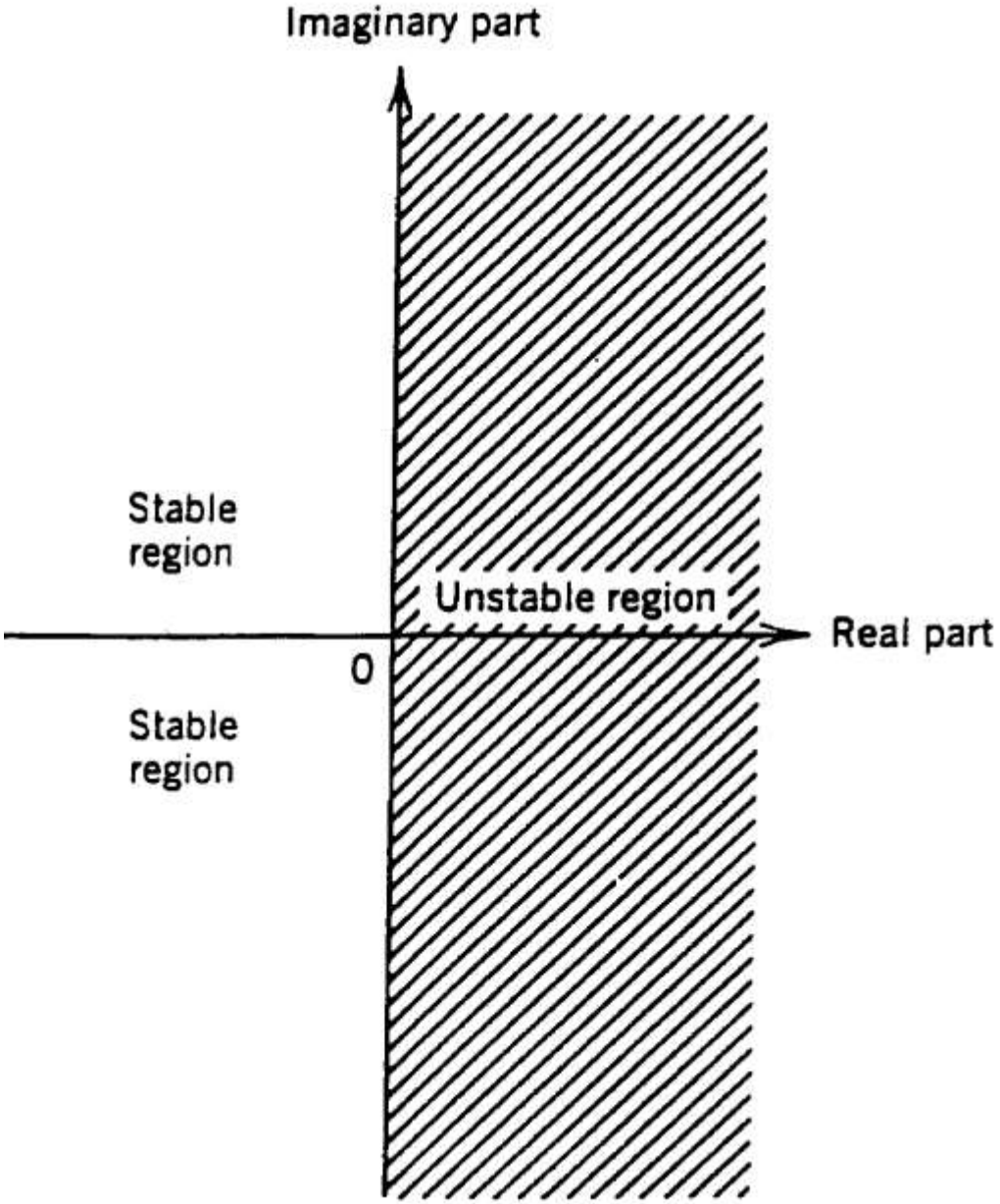

Figure 11.25
Stability regions in the complex plane for roots of the charact-eristic equation.

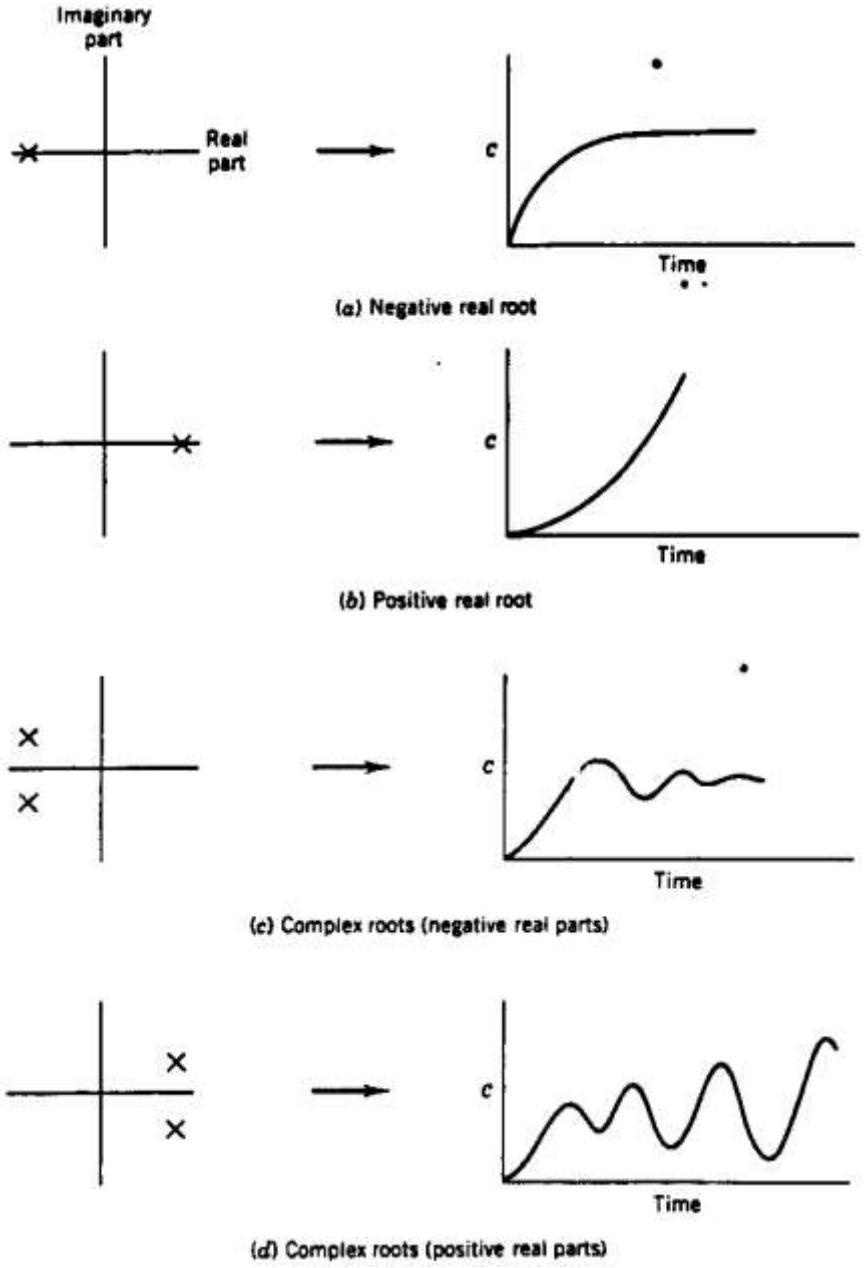

Figure 11.26 Contributions of characteristic equation roots to closed-loop response.

Solution

The characteristic equation for this system is

$$s + 0.2K_c - 1 = 0 \qquad (11\text{-}92)$$

Which has the single root, $s = -1 + 0.2K_c$. Thus, the stability requirement is that $K_c < 5$. This example illustrates the important fact that feedback control can be used to stabilize a process that is not stable without control.

Routh Stability Criterion

The Routh stability criterion is based on a characteristic equation that has the form

$$a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0 \qquad (11\text{-}93)$$

Routh array:

Row				
1	a_n	a_{n-2}	a_{n-4}	$\cdots$
2	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
3	b_1	b_2	b_3	$\cdots$
4	c_1	c_2	$\cdots$	
$\vdots$	$\vdots$			
$n+1$	z_1			

where:

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \qquad (11\text{-}94)$$

$$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}} \qquad (11\text{-}95)$$

$\vdots$

and:

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1} \qquad (11\text{-}96)$$

$$c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1} \qquad (11\text{-}97)$$

$$\vdots$$

Routh Stability Criterion:

A necessary and sufficient condition for all roots of the characteristic equation in Eq. 11-93 to have negative real parts is that all of the elements in the left column of the Routh array are positive.

Example 11.9

Determine the stability of a system that has the characteristic equation

$$s^4 + 5s^3 + 3s^2 + 1 = 0 \qquad (11\text{-}98)$$

Solution

Because the *s* term is missing, its coefficient is zero. Thus, the system is unstable. Recall that a necessary condition for stability is that all of the coefficients in the characteristic equation must be positive.

Example 11.10

Find the values of controller gain K_c that make the feedback control system of Eq. 11.4 stable.

Solution

From Eq. 11-76, the characteristic equation is

$$10s^3 + 17s^2 + 8s + 1 + K_c = 0 \qquad (11\text{-}99)$$

All coefficients are positive provided that $1 + K_c > 0$ or $K_c < -1$. The Routh array is

10	8
17	$1 + K_c$
b_1	b_2
c_1	

To have a stable system, each element in the left column of the Routh array must be positive. Element b_1 will be positive if $K_c < 7.41/0.588 = 12.6$. Similarly, c_1 will be positive if $K_c > -1$. Thus, we conclude that the system will be stable if

$$-1 < K_c < 12.6 \qquad (11\text{-}100)$$

Direct Substitution Method

- The imaginary axis divides the complex plane into stable and unstable regions for the roots of characteristic equation, as indicated in Fig. 11.26.
- On the imaginary axis, the real part of s is zero, and thus we can write $s=j\omega$. Substituting $s=j\omega$ into the characteristic equation allows us to find a stability limit such as the maximum value of K_c.
- As the gain K_c is increased, the roots of the characteristic equation cross the imaginary axis when $K_c = K_{cm}$.

Example 11.12

Use the direct substitution method to determine K_{cm} for the system with the characteristic equation given by Eq. 11-99.

Solution

Substitute $s = j\omega$ and $K_c = K_{cm}$ into Eq. 11-99:

$$-10j\omega^3 - 17\omega^2 + 8j\omega + 1 + K_{cm} = 0$$

or

$$\left(1 + K_{cm} - 17\omega^2\right) + j\left(8\omega - 10\omega^3\right) = 0 \tag{11-105}$$

Equation 11-105 is satisfied if both the real and imaginary parts are identically zero:

$$1+K_{cm}-17\omega^2=0 \qquad \text{(11-106a)}$$

$$8\omega-10\omega^3=\omega\left(8-10\omega^2\right)=0 \qquad \text{(11-106b)}$$

Therefore,

$$\omega^2=0.8\Rightarrow\omega=\pm0.894 \qquad \text{(11-107)}$$

and from (11-106a),

$$K_{cm}=12.6$$

Root Locus Diagrams

Example 11.13

Consider a feedback control system that has the open-loop transfer function,

$$G_{OL}(s) = \frac{4K_c}{(s+1)(s+2)(s+3)} \qquad (11\text{-}108)$$

Plot the root locus diagram for $0 \le K_c \le 20$.

Solution

The characteristic equation is $1 + G_{OL} = 0$ or

$$(s+1)(s+2)(s+3) + 4K_c = 0 \qquad (11\text{-}109)$$

- The root locus diagram in Fig. 11.27 shows how the three roots of this characteristic equation vary with K_c.
- When $K_c = 0$, the roots are merely the poles of the open-loop transfer function, -1, -2, and -3.

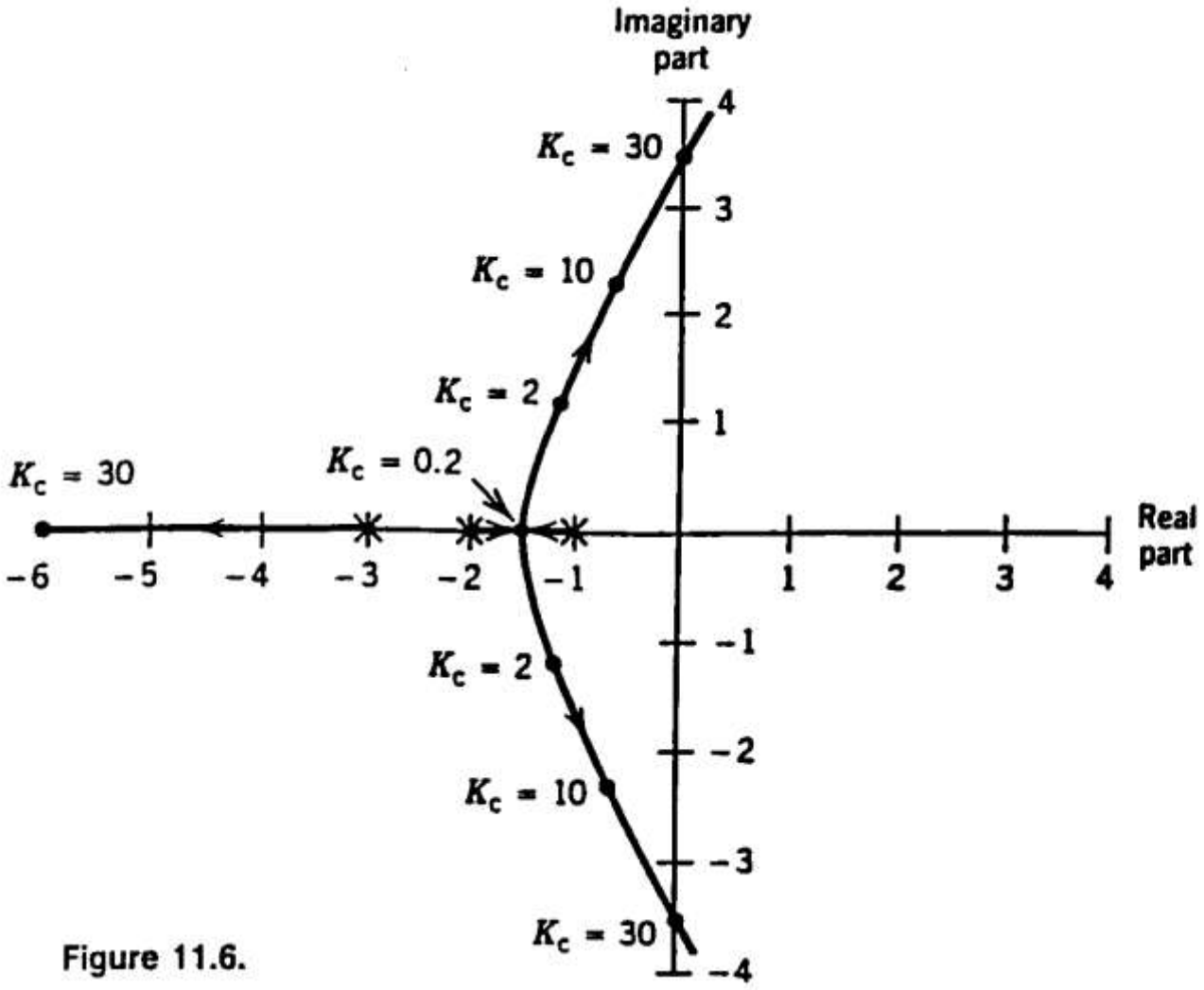

Figure 11.6.

Figure 11.27 Root locus diagram for third-order system. X denotes an open-loop pole. Dots denote locations of the closed-loop poles for different values of K_c. Arrows indicate change of pole locations as K_c increases.

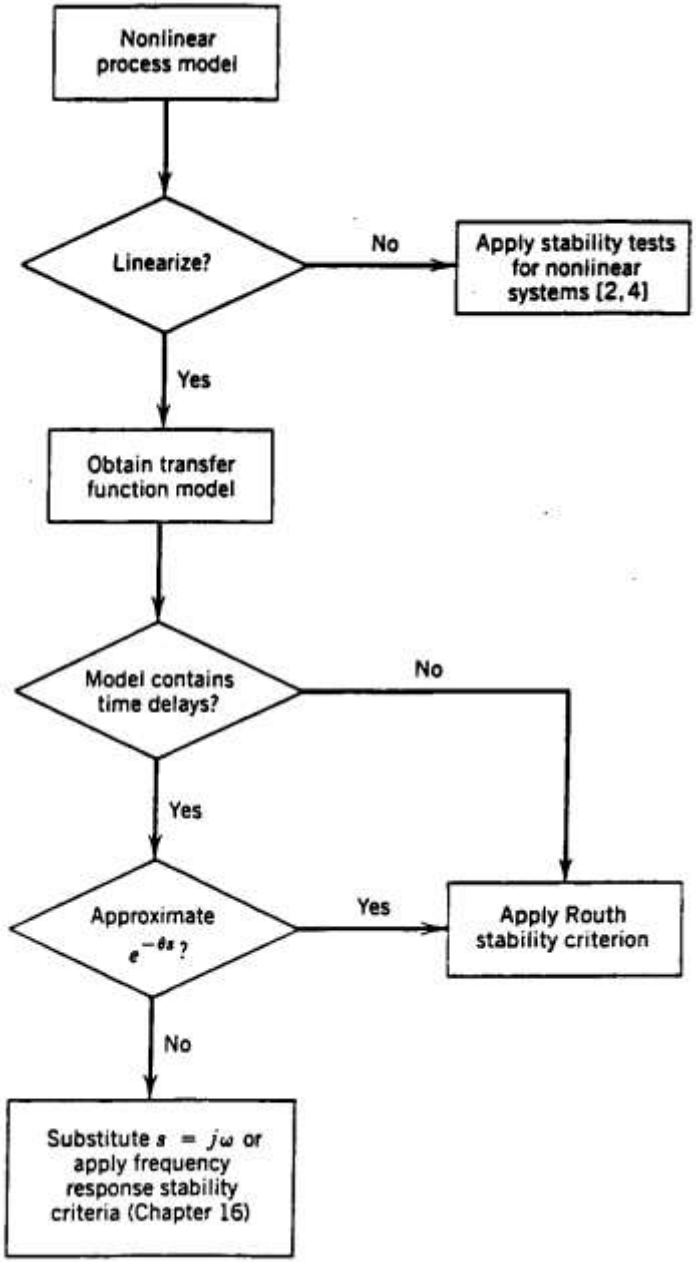

Figure 11.29. Flowchart for performing a stability analysis.