Chapter 13

Frequency Response Analysis

Sinusoidal Forcing of a First-Order Process

For a first-order transfer function with gain K and time constant τ, the response to a general sinusoidal input, $x(t) = A\sin\omega t$ is:

$$y(t) = \frac{KA}{\omega^2\tau^2+1}\left(\omega\tau e^{-t/\tau} - \omega\tau\cos\omega t + \sin\omega t\right) \qquad (5\text{-}25)$$

Note that *y(t)* and *x(t)* are in deviation form. The *long-time response*, $y_l(t)$, can be written as:

$$y_\ell(t) = \frac{KA}{\sqrt{\omega^2\tau^2+1}}\sin(\omega t+\varphi) \text{ for } t\to\infty \qquad (13\text{-}1)$$

where:

$$\varphi = -\tan^{-1}(\omega\tau)$$

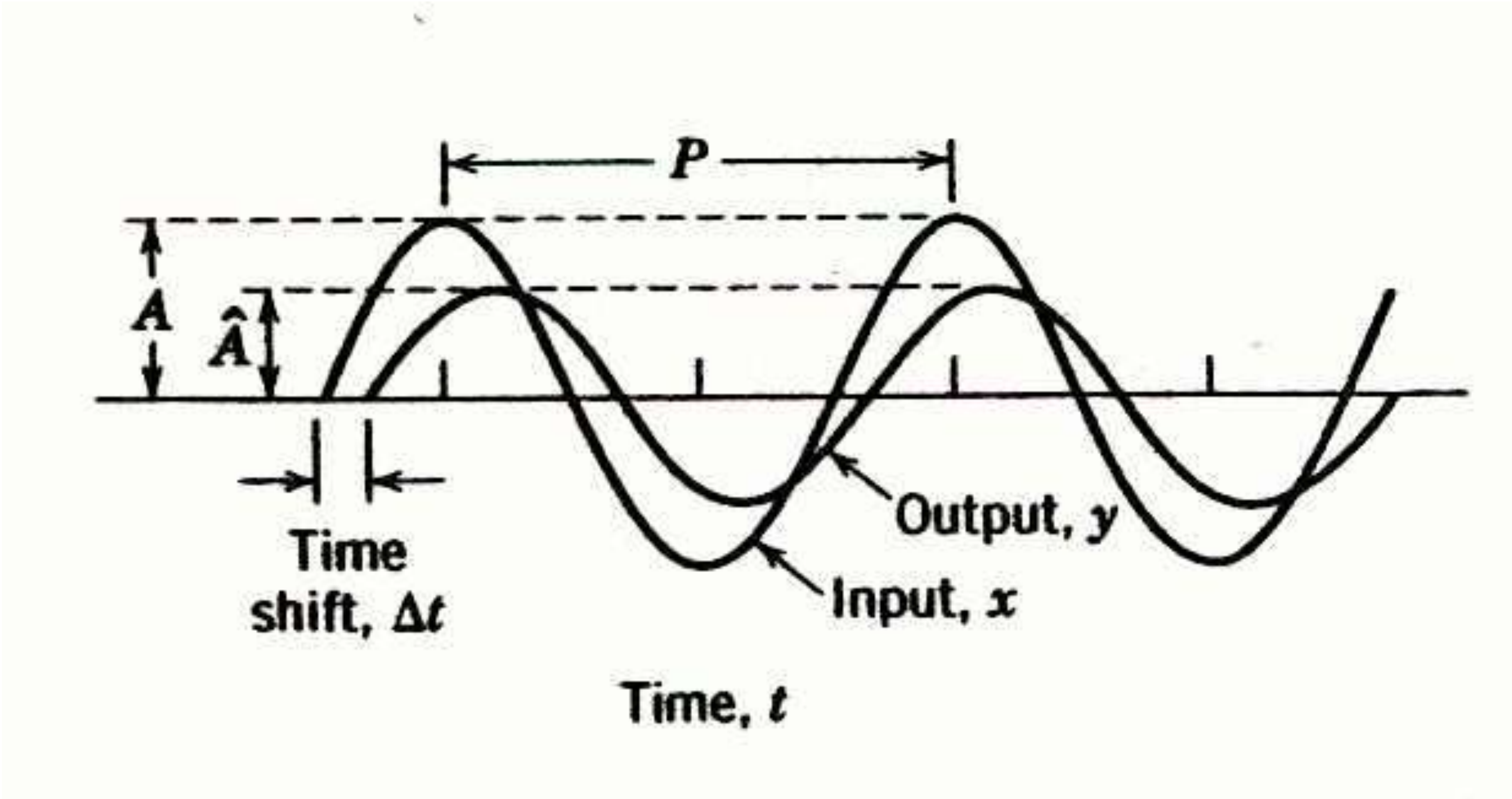

Figure 13.1 Attenuation and time shift between input and output sine waves (K= 1). The phase angle φ of the output signal is given by $\varphi = -\text{Time shift} / P \times 360^\circ$, where Δt is the (period) shift and P is the period of oscillation.

Frequency Response Characteristics of a First-Order Process

For $x(t) = A\sin\omega t,\ \ y_\ell(t) = \hat{A}\sin(\omega t + \varphi)$ as $t \to \infty$ where :

$$\hat{A} = \frac{KA}{\sqrt{\omega^2\tau^2 + 1}} \quad \text{and} \quad \varphi = -\tan^{-1}(\omega\tau)$$

1. The output signal is a sinusoid that has the same frequency, ω, as the input.signal, $x(t) = A\sin\omega t$.

2. The amplitude of the output signal, $\hat{A}$, is a function of the frequency ω and the input amplitude, *A*:

$$\hat{A} = \frac{KA}{\sqrt{\omega^2\tau^2 + 1}} \qquad (13\text{-}2)$$

3. The output has a phase shift, φ, relative to the input. The amount of phase shift depends on ω.

Dividing both sides of (13-2) by the input signal amplitude *A* yields the *amplitude ratio* (AR)

$$\text{AR} = \frac{\hat{A}}{A} = \frac{K}{\sqrt{\omega^2\tau^2+1}} \tag{13-3a}$$

which can, in turn, be divided by the process gain to yield the *normalized amplitude ratio* (AR_N)

$$\text{AR}_\text{N} = \frac{1}{\sqrt{\omega^2\tau^2+1}} \tag{13-3b}$$

Shortcut Method for Finding the Frequency Response

The shortcut method consists of the following steps:

Step 1. Set $s = j\omega$ in $G(s)$ to obtain $G(j\omega)$.

Step 2. Rationalize $G(j\omega)$; We want to express it in the form.

$$G(j\omega) = R + jI$$

where R and I are functions of ω. Simplify $G(j\omega)$ by multiplying the numerator and denominator by the complex conjugate of the denominator.

Step 3. The amplitude ratio and phase angle of $G(s)$ are given by:

Memorize ⇒

$$\text{AR} = \sqrt{R^2 + I^2}$$

$$\varphi = \tan^{-1}(R/I)$$

Example 13.1

Find the frequency response of a first-order system, with

$$G(s) = \frac{1}{\tau s + 1} \tag{13-16}$$

Solution

First, substitute $s = j\omega$ in the transfer function

$$G(j\omega) = \frac{1}{\tau j\omega + 1} = \frac{1}{j\omega\tau + 1} \tag{13-17}$$

Then multiply both numerator and denominator by the complex conjugate of the denominator, that is, $-j\omega\tau + 1$

$$G(j\omega) = \frac{-j\omega\tau + 1}{(j\omega\tau + 1)(-j\omega\tau + 1)} = \frac{-j\omega\tau + 1}{\omega^2\tau^2 + 1}$$

$$= \frac{1}{\omega^2\tau^2 + 1} + j\frac{(-\omega\tau)}{\omega^2\tau^2 + 1} = R + jI \tag{13-18}$$

where: $$R = \frac{1}{\omega^2\tau^2 + 1} \qquad (13\text{-}19a)$$

$$I = \frac{-\omega\tau}{\omega^2\tau^2 + 1} \qquad (13\text{-}19b)$$

From Step 3 of the Shortcut Method,

$$\text{AR} = \sqrt{R^2 + I^2} = \sqrt{\left(\frac{1}{\omega^2\tau^2 + 1}\right)^2 + \left(\frac{-\omega\tau}{\omega^2\tau^2 + 1}\right)^2}$$

or

$$\text{AR} = \sqrt{\frac{\left(1 + \omega^2\tau^2\right)}{\left(\omega^2\tau^2 + 1\right)^2}} = \frac{1}{\sqrt{\omega^2\tau^2 + 1}} \qquad (13\text{-}20a)$$

Also,

$$\varphi = \tan^{-1}\left(\frac{I}{R}\right) = \tan^{-1}\left(-\omega\tau\right) = -\tan^{-1}\left(\omega\tau\right) \qquad (13\text{-}20b)$$

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Complex Transfer Functions

Consider a complex transfer $G(s)$,

$$G(s) = \frac{G_a(s)G_b(s)G_c(s)\cdots}{G_1(s)G_2(s)G_3(s)\cdots} \qquad (13\text{-}22)$$

Substitute $s=j\omega$,

$$G(j\omega) = \frac{G_a(j\omega)G_b(j\omega)G_c(j\omega)\cdots}{G_1(j\omega)G_2(j\omega)G_3(j\omega)\cdots} \qquad (13\text{-}23)$$

From complex variable theory, we can express the magnitude and angle of $G(j\omega)$ as follows:

$$|G(j\omega)| = \frac{|G_a(j\omega)||G_b(j\omega)||G_c(j\omega)|\cdots}{|G_1(j\omega)||G_2(j\omega)||G_3(j\omega)|\cdots} \qquad (13\text{-}24a)$$

$$\angle G(j\omega) = \angle G_a(j\omega) + \angle G_b(j\omega) + \angle G_c(j\omega) + \cdots$$
$$-[\angle G_1(j\omega) + \angle G_2(j\omega) + \angle G_3(j\omega) + \cdots] \qquad (13\text{-}24b)$$

Bode Diagrams

- A special graph, called the *Bode diagram* or *Bode plot*, provides a convenient display of the frequency response characteristics of a transfer function model. It consists of plots of AR and φ as a function of ω.
- Ordinarily, ω is expressed in units of radians/time.

Bode Plot of A First-order System

Recall:

$$AR_N = \frac{1}{\sqrt{\omega^2\tau^2+1}} \quad \text{and} \quad \varphi = -\tan^{-1}(\omega\tau)$$

- **At low frequencies** $(\omega \to 0 \text{ and } \omega\tau \ll 1)$:

$$AR_N = 1 \quad \text{and} \quad \varphi = 0$$

- **At high frequencies** $(\omega \to 0 \text{ and } \omega\tau \gg 1)$:

$$AR_N = 1/\omega\tau \quad \text{and} \quad \varphi = -90^\circ$$

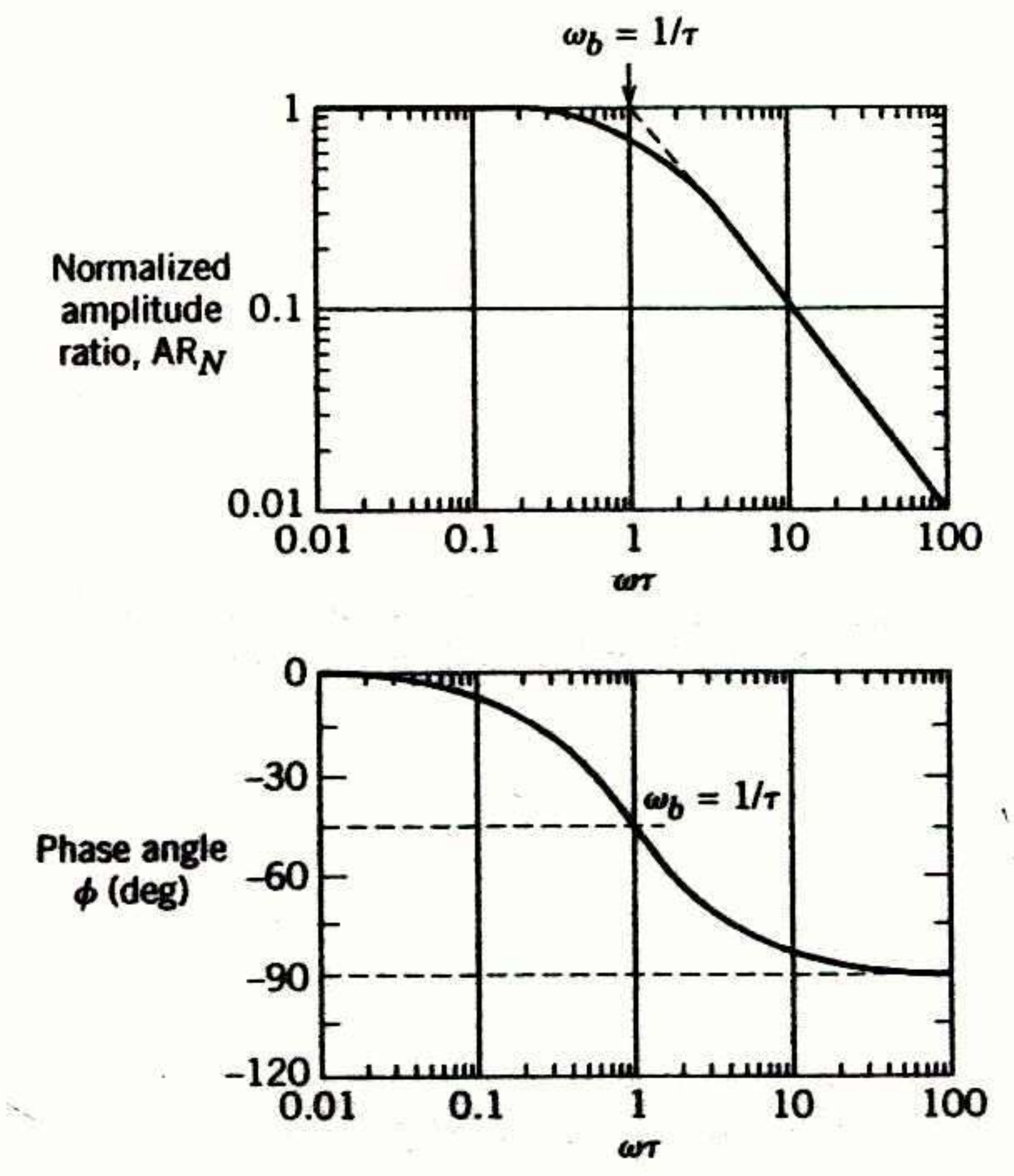

Figure 13.2 Bode diagram for a first-order process.

- Note that the asymptotes intersect at $\omega = \omega_b = 1/\tau$, known as the *break frequency* or *corner frequency*. Here the value of AR_N from (13-21) is:

$$\text{AR}_\text{N}\left(\omega = \omega_b\right) = \frac{1}{\sqrt{1+1}} = 0.707 \qquad (13\text{-}30)$$

- Some books and software defined AR differently, in terms of *decibels*. The amplitude ratio in decibels AR_d is defined as

$$\text{AR}_\text{d} = 20 \ \log \ \text{AR} \qquad (13\text{-}33)$$

Integrating Elements

The transfer function for an integrating element was given in Chapter 5:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{s} \qquad (5\text{-}34)$$

$$\text{AR} = |G(j\omega)| = \left|\frac{K}{j\omega}\right| = \frac{K}{\omega} \qquad (13\text{-}34)$$

$$\varphi = \angle G(j\omega) = \angle K - \angle(\infty) = -90^\circ \qquad (13\text{-}35)$$

Second-Order Process

A general transfer function that describes any underdamped, critically damped, or overdamped second-order system is

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1} \qquad (13\text{-}40)$$

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Substituting $s = j\omega$ and rearranging yields:

$$\text{AR} = \frac{K}{\sqrt{\left(1-\omega^2\tau^2\right)^2 + \left(2\omega\tau\zeta\right)^2}} \qquad \text{(13-41a)}$$

$$\varphi = \tan^{-1}\left[\frac{-2\omega\tau\zeta}{1-\omega^2\tau^2}\right] \qquad \text{(13-41b)}$$

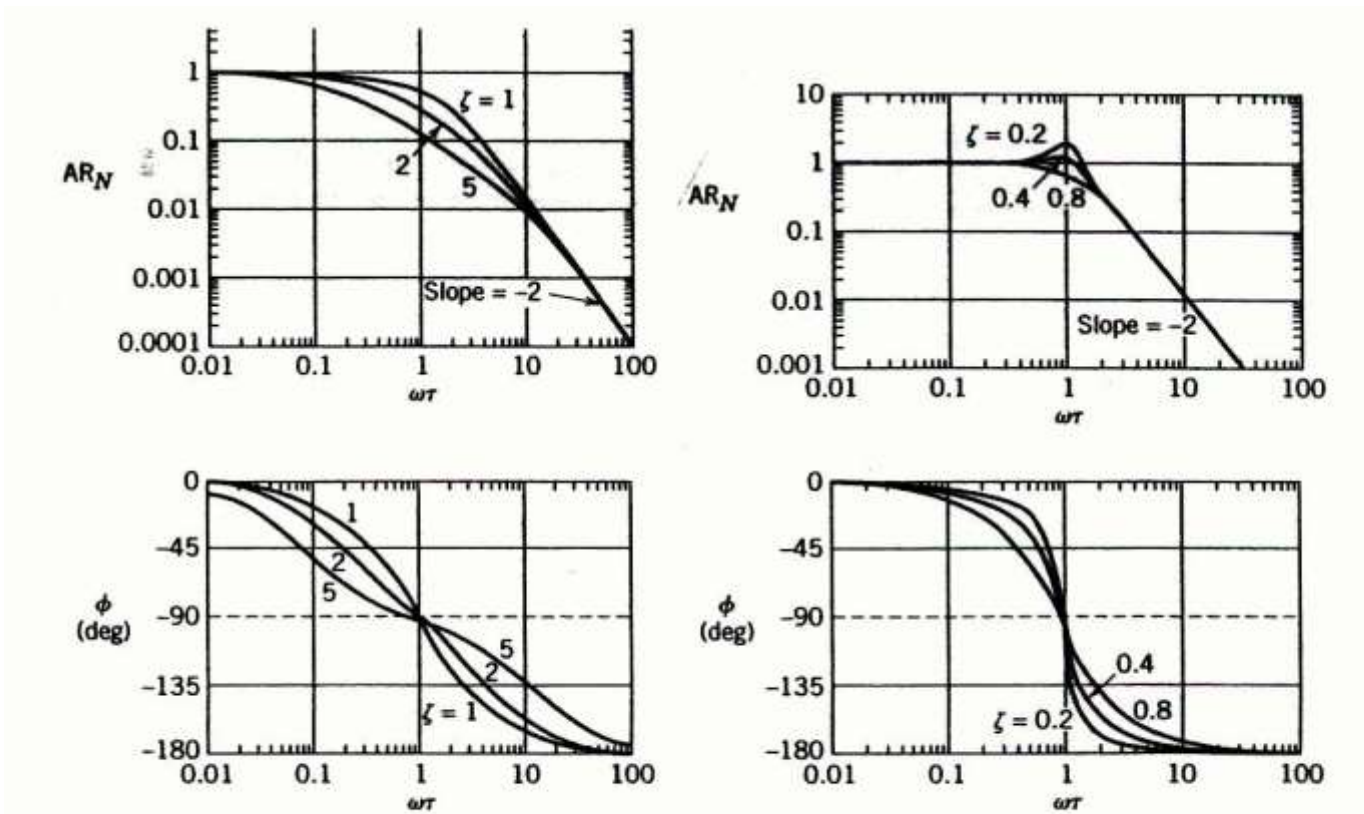

Figure 13.3 Bode diagrams for second-order processes.

Time Delay

Its frequency response characteristics can be obtained by substituting $s = j\omega$,

$$G(j\omega) = e^{-j\omega\theta} \qquad (13\text{-}53)$$

which can be written in rational form by substitution of the Euler identity,

$$G(j\omega) = e^{-j\omega\theta} = \cos\omega\theta - j\sin\omega\theta \qquad (13\text{-}54)$$

From (13-54)

$$\text{AR} = |G(j\omega)| = \sqrt{\cos^2\omega\theta + \sin^2\omega\theta} = 1 \qquad (13\text{-}55)$$

$$\varphi = \angle G(j\omega) = \tan^{-1}\left(-\frac{\sin\omega\theta}{\cos\omega\theta}\right)$$

or

$$\varphi = -\omega\theta \qquad (13\text{-}56)$$

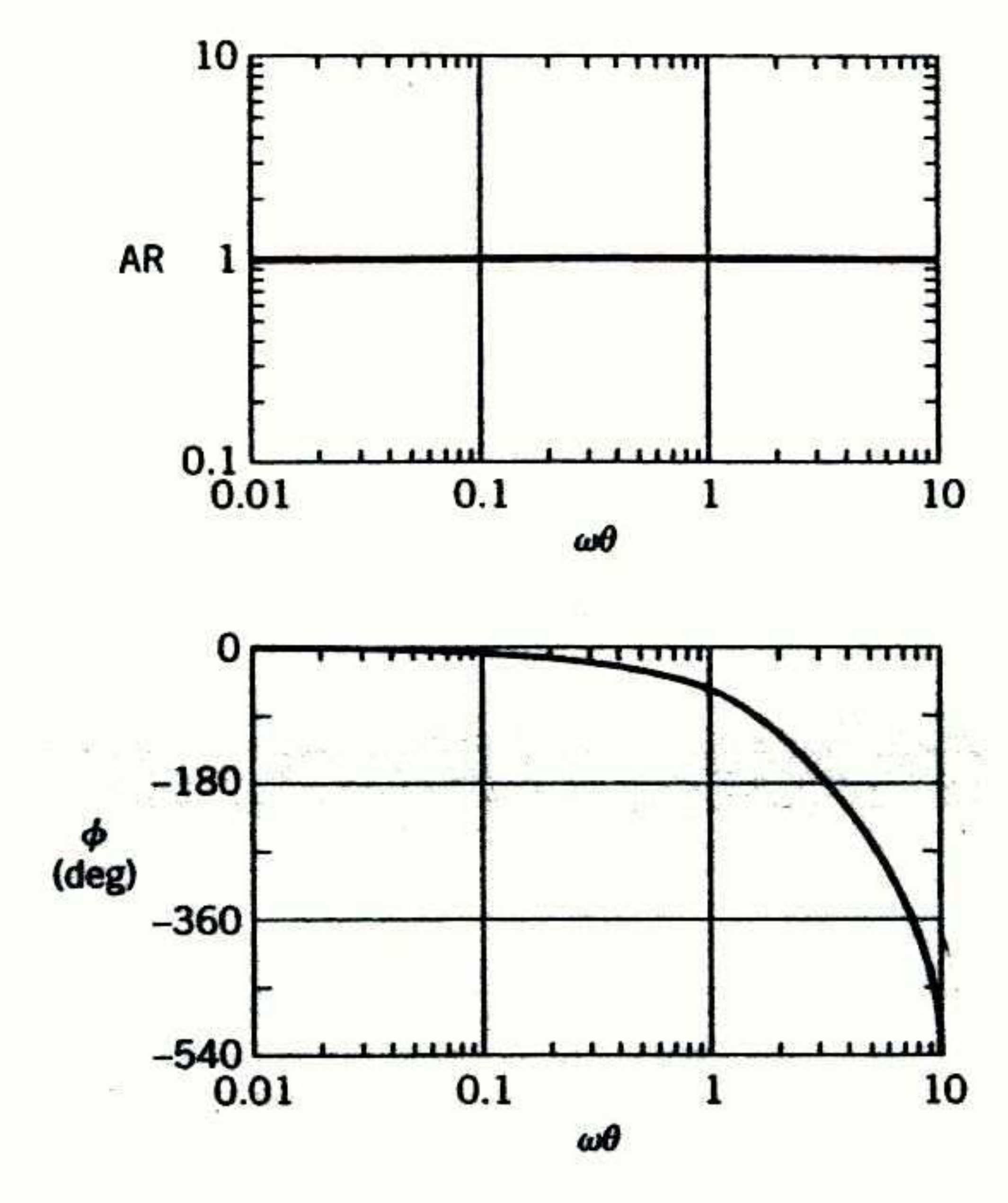

Figure 13.6 Bode diagram for a time delay, $e^{-\theta s}$.

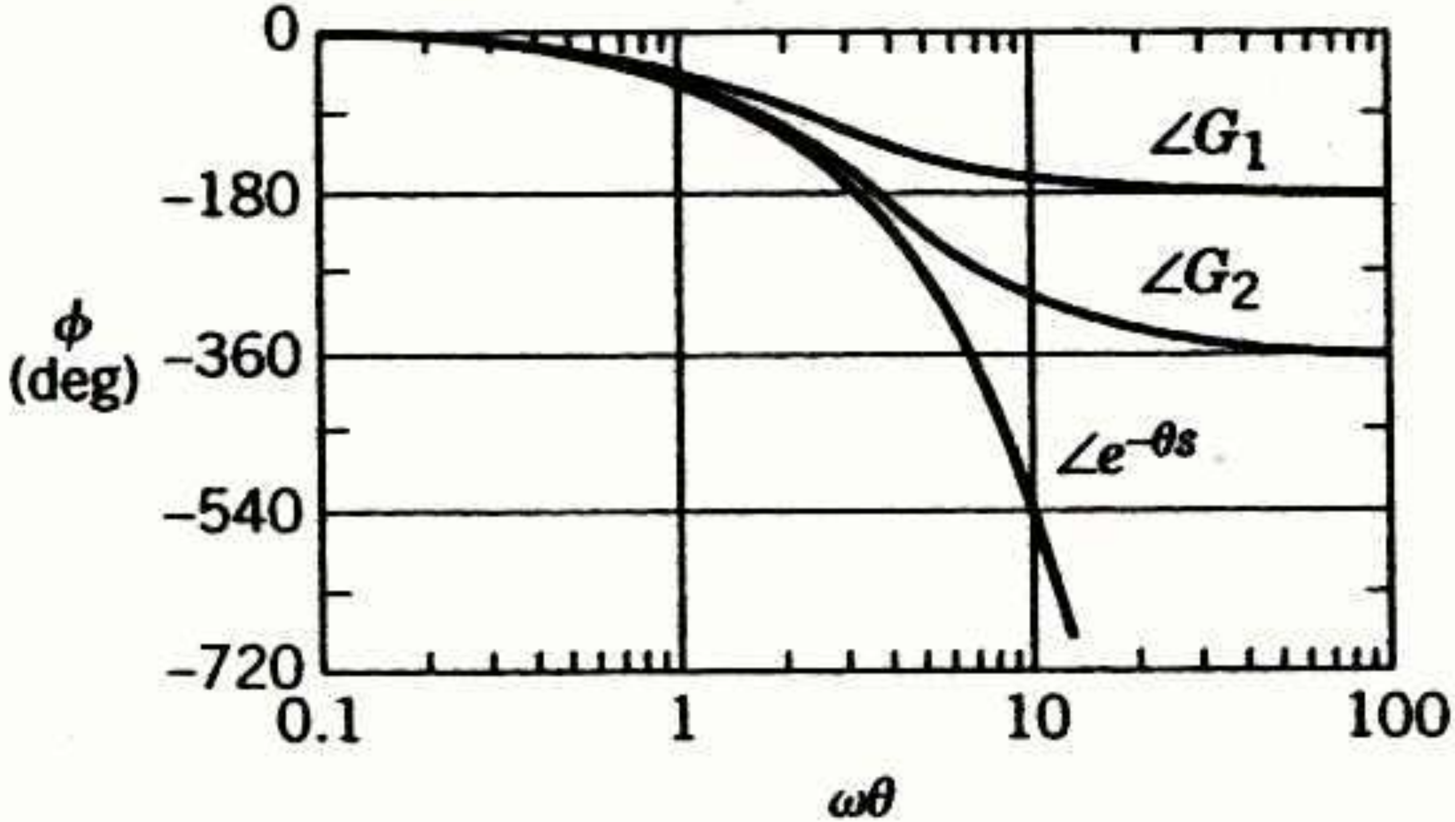

Figure 13.7 Phase angle plots for $e^{-\theta s}$ and for the 1/1 and 2/2 Padé approximations (G_1 is 1/1; G_2 is 2/2).

Process Zeros

Consider a process zero term,

$$G(s) = K(s\tau + 1)$$

Substituting $s = j\omega$ gives

$$G(j\omega) = K(j\omega\tau + 1)$$

Thus:

$$\text{AR} = |G(j\omega)| = K\sqrt{\omega^2\tau^2 + 1}$$

$$\varphi = \angle G(j\omega) = +\tan^{-1}(\omega\tau)$$

Note: In general, a multiplicative constant (e.g., K) changes the AR by a factor of K without affecting φ.

Frequency Response Characteristics of Feedback Controllers

Proportional Controller. Consider a proportional controller with positive gain

$$G_c(s) = K_c \qquad (13\text{-}57)$$

In this case $|G_c(j\omega)| = K_c$, which is independent of ω. Therefore,

$$\text{AR}_c = K_c \qquad (13\text{-}58)$$

and

$$\varphi_c = 0^\circ \qquad (13\text{-}59)$$

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Proportional-Integral Controller. A proportional-integral (PI) controller has the transfer function (cf. Eq. 8-9),

$$G_c(s) = K_c\left(1+\frac{1}{\tau_I s}\right) = K_c\left(\frac{\tau_I s+1}{\tau_I s}\right) \qquad (13\text{-}60)$$

Substitute s=jω:

$$G_c(j\omega) = K_c\left(1+\frac{1}{\tau_I j\omega}\right) = K_c\left(\frac{j\omega\tau_I+1}{j\omega\tau_I}\right) = K_c\left(1-\frac{1}{\tau_I\omega}j\right)$$

Thus, the amplitude ratio and phase angle are:

$$\text{AR}_c = \left|G_c(j\omega)\right| = K_c\sqrt{1+\frac{1}{(\omega\tau_I)^2}} = K_c\frac{\sqrt{(\omega\tau_I)^2+1}}{\omega\tau_I} \qquad (13\text{-}62)$$

$$\varphi_c = \angle G_c(j\omega) = \tan^{-1}(-1/\omega\tau_I) = \tan^{-1}(\omega\tau_I) - 90^\circ \qquad (13\text{-}63)$$

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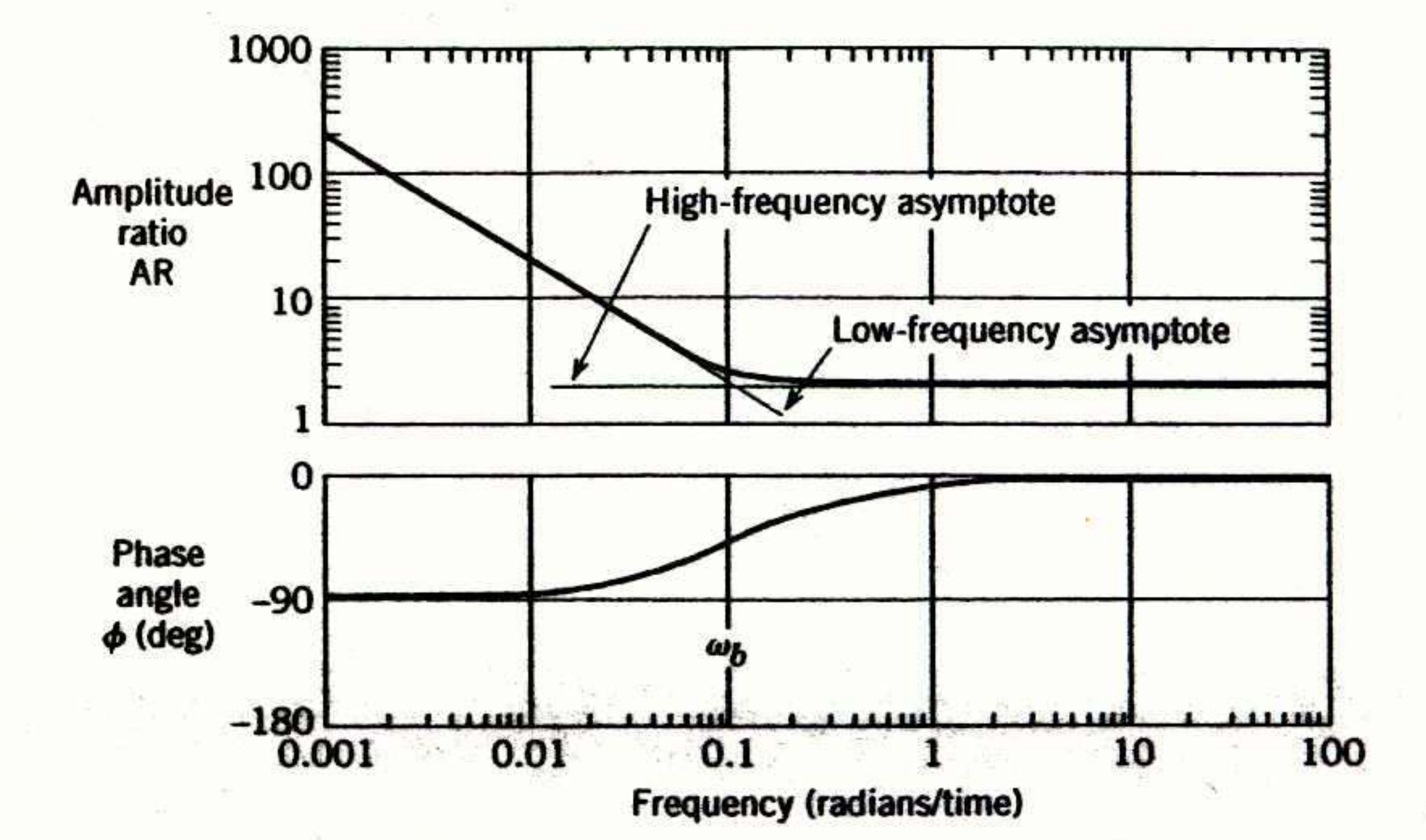

Figure 13.9 Bode plot of a PI controller, $G_c(s) = 2\left(\frac{10s+1}{10s}\right)$

Ideal Proportional-Derivative Controller. For the ideal proportional-derivative (PD) controller (cf. Eq. 8-11)

$$G_c(s) = K_c(1 + \tau_D s) \qquad (13\text{-}64)$$

The frequency response characteristics are similar to those of a LHP zero:

$$\mathrm{AR}_c = K_c\sqrt{(\omega\tau_D)^2 + 1} \qquad (13\text{-}65)$$

$$\varphi = \tan^{-1}(\omega\tau_D) \qquad (13\text{-}66)$$

Proportional-Derivative Controller with Filter. The PD controller is most often realized by the transfer function

$$G_c(s) = K_c\left(\frac{\tau_D s + 1}{\alpha\tau_D s + 1}\right) \qquad (13\text{-}67)$$

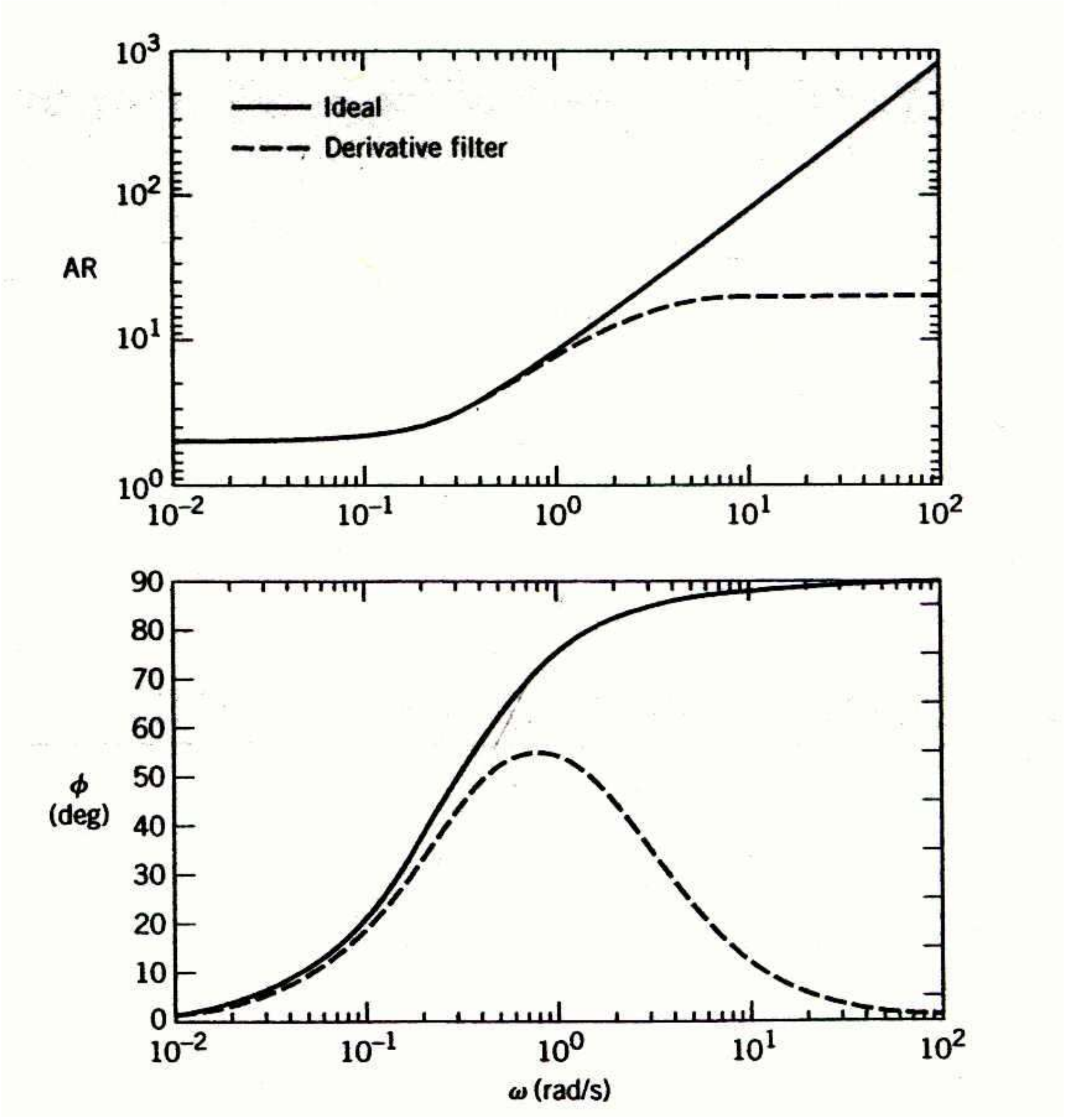

Figure 13.10 Bode plots of an ideal PD controller and a PD controller with derivative filter.

Idea: $G_c(s) = 2(4s+1)$

With Derivative Filter:

$$G_c(s) = 2\left(\frac{4s+1}{0.4s+1}\right)$$

PID Controller Forms

Parallel PID Controller. The simplest form in Ch. 8 is

$$G_c(s) = K_c\left(1 + \frac{1}{\tau_1 s} + \tau_D s\right)$$

Series PID Controller. The simplest version of the series PID controller is

$$G_c(s) = K_c\left(\frac{\tau_1 s + 1}{\tau_1 s}\right)(\tau_D s + 1) \qquad (13\text{-}73)$$

Series PID Controller with a Derivative Filter.

$$G_c(s) = K_c\left(\frac{\tau_1 s + 1}{\tau_1 s}\right)\left(\frac{\tau_D s + 1}{\alpha \tau_D s + 1}\right)$$

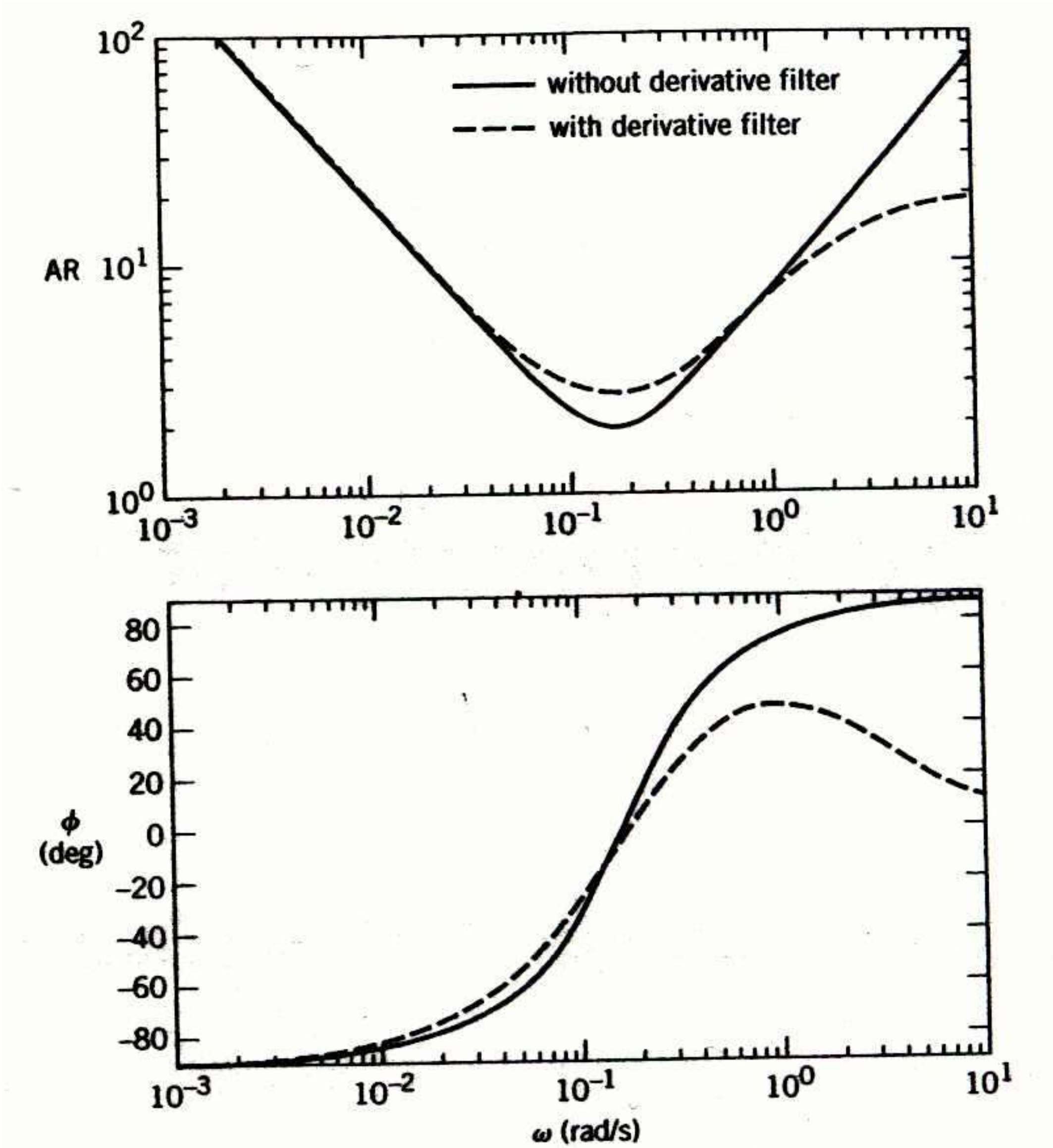

Figure 13.11 Bode plots of ideal parallel PID controller and series PID controller with derivative filter (α = 1).

Idea parallel:

$$G_c(s) = 2\left(1 + \frac{1}{10s} + 4s\right)$$

Series with Derivative Filter:

$$G_c(s) = 2\left(\frac{10s+1}{10s}\right)\left(\frac{4s+1}{0.4s+1}\right)$$

Nyquist Diagrams

Consider the transfer function

$$G(s) = \frac{1}{2s+1} \qquad (13\text{-}76)$$

with

$$\text{AR} = |G(j\omega)| = \frac{1}{\sqrt{(2\omega)^2+1}} \qquad (13\text{-}77a)$$

and

$$\varphi = \angle G(j\omega) = -\tan^{-1}(2\omega) \qquad (13\text{-}77b)$$

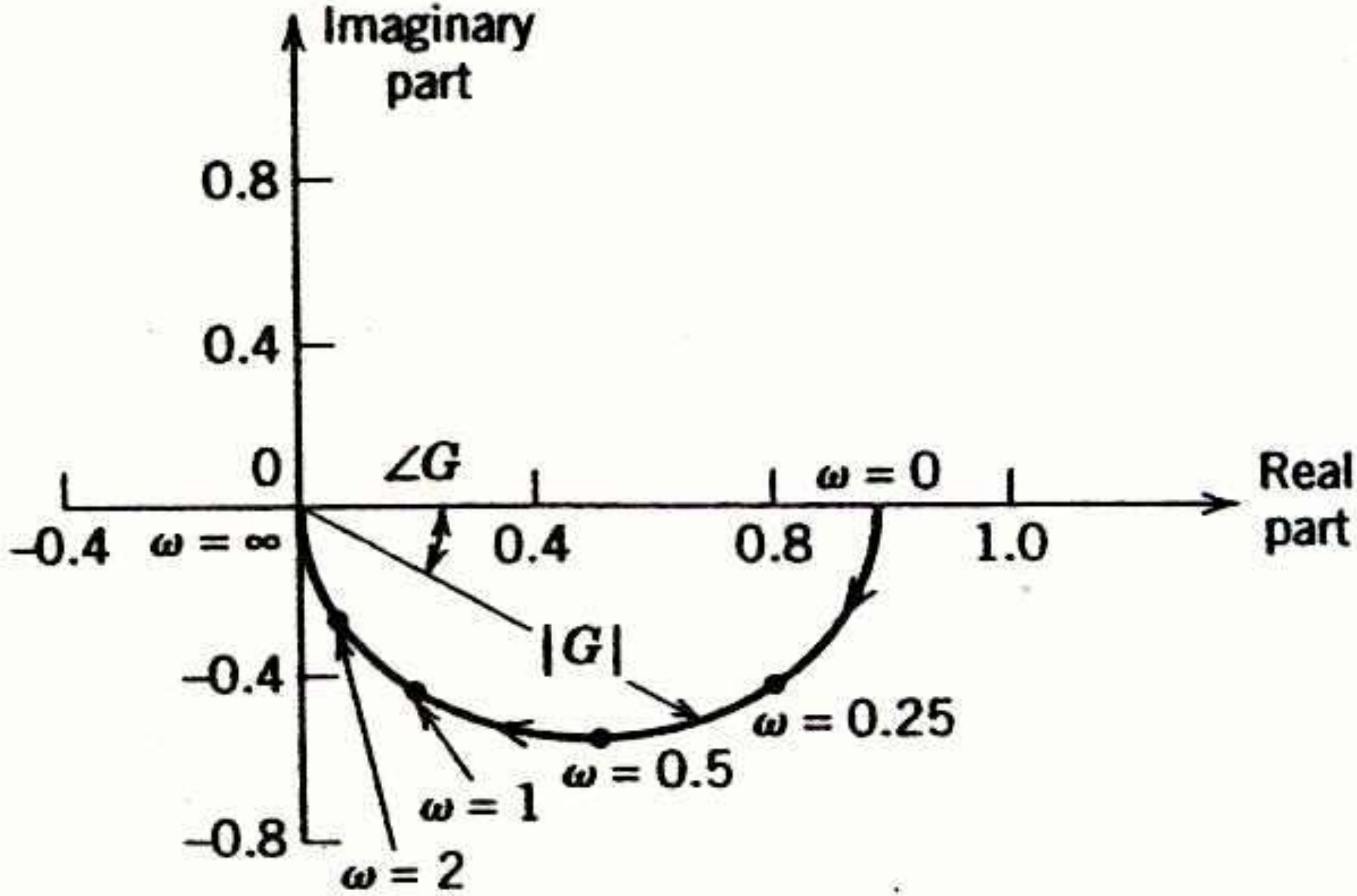

Figure 13.12 The Nyquist diagram for $G(s) = 1/(2s + 1)$ plotting $\mathrm{Re}\big(G(j\omega)\big)$ and $\mathrm{Im}\big(G(j\omega)\big)$.

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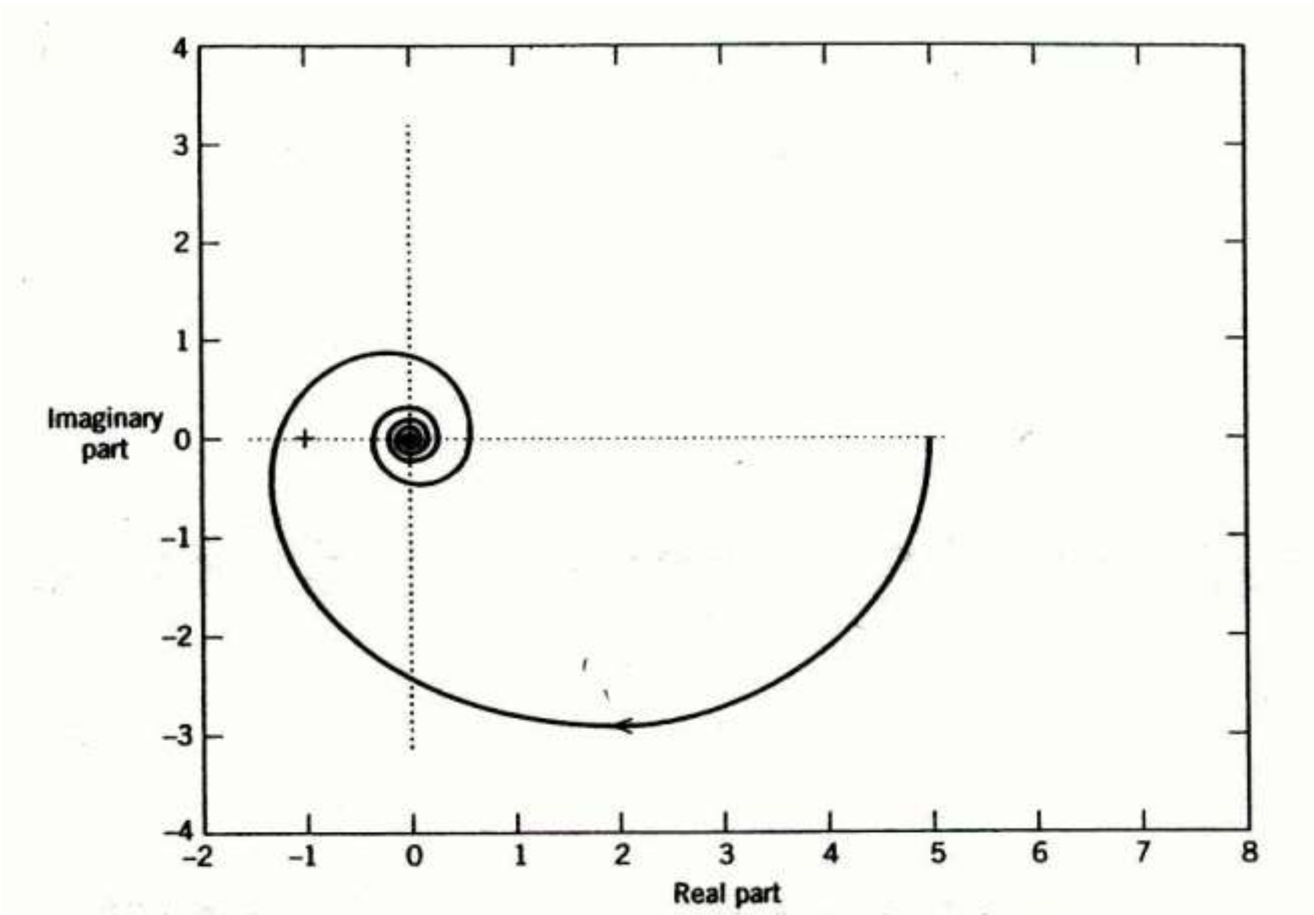

Figure 13.13 The Nyquist diagram for the transfer function in Example 13.5:

$$G(s) = \frac{5(8s+1)e^{-6s}}{(20s+1)(4s+1)}$$