Process Heat Transfer

Lec 4: Heat Conduction Equation

Steady-State Conduction with Thermal Energy Generation

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Implications of Energy Generation

- Involves a local (volumetric) source of thermal energy due to conversion from another form of energy in a conducting medium.
- The source may be uniformly distributed, as in the conversion from electrical to thermal energy (Ohmic heating):

Volumetric generation rate $\dot{q} \equiv \frac{\dot{E}_g}{V} = \frac{I^2 R_e}{V}$ (W/m^3)

or it may be non-uniformly distributed, as in the absorption of radiation passing through a semi-transparent medium.

- Generation affects the temperature distribution in the medium and causes the heat rate to vary with location, thereby precluding inclusion of the medium in a thermal circuit.

The Plane Wall

- Consider one-dimensional, steady-state conduction in a plane wall of constant k, uniform generation, and asymmetric surface conditions:

- Heat Equation: $\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{q} = 0$

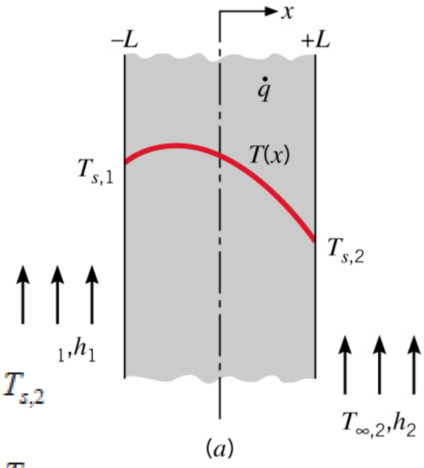
$$\Rightarrow \quad \frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

- General Solution: $T = -\frac{\dot{q}}{2k}x^2 + C_1x + C_2$

- Boundary conditions $\quad T(-L) = T_{s,1} \quad$ and $\quad T(L) = T_{s,2}$

$$\Rightarrow \quad C_1 = \frac{T_{s,2} - T_{s,1}}{2L} \quad \text{and} \quad C_2 = \frac{\dot{q}}{2k}L^2 + \frac{T_{s,1} + T_{s,2}}{2}$$

$$\Rightarrow \quad T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,1} + T_{s,2}}{2}$$



The Plane Wall

- Is the heat flux q'' independent of x? Answer is required

Symmetric Surface Conditions or One Surface Insulated:

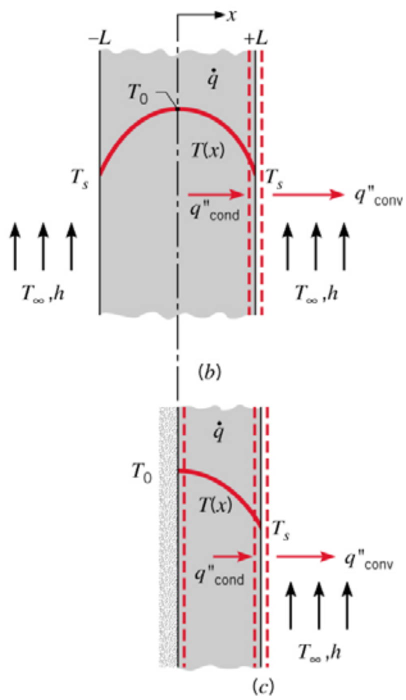
- Boundary conditions

$$\text{at } x = \pm L \qquad T_{s,1} = T_{s,2} \equiv T_s$$

$$\Rightarrow \quad T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s$$

- At the mid-plane, x =0.0
- The maximum temperature exists

$$T(0) \equiv T_0 = \frac{\dot{q}L^2}{2k} + T_s \qquad \text{and} \qquad (dT/dx)_{x=0} = 0.$$



The Plane Wall

- Hence, the temperature distribution, may be expressed as

$$\frac{T(x) - T_0}{T_s - T_0} = \left(\frac{x}{L}\right)^2$$

- If the temperature of an adjoining fluid T_∞, is known and not T_s

- Surface energy balance → $-k\frac{dT}{dx}\bigg|_{x=L} = h(T_s - T_\infty)$ ⟹ $T_s = T_\infty + \frac{\dot{q}L}{h}$

- Overall energy balance on the wall → $\dot{E}_g = \dot{E}_{out}$ ⟹ $\dot{q}L = h(T_s - T_\infty)$

- How do we determine the heat rate at $x = L$?

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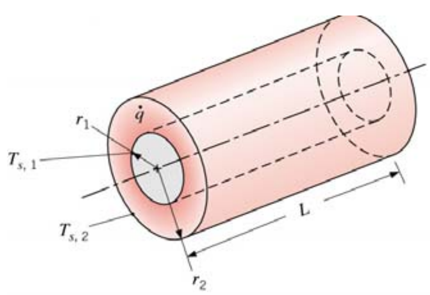
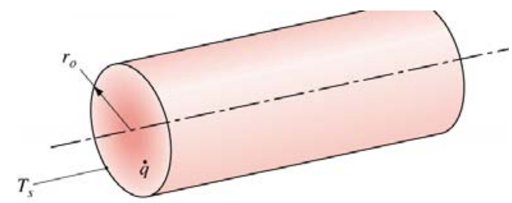
Cylindrical Systems

Cylindrical (Tube) Wall



Solid Cylinder (Circular Rod)



- Heat Equations: $\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{q}}{k} = 0$

Separating variables and assuming uniform generation and constant *k*

$$r\frac{dT}{dr} = -\frac{\dot{q}}{2k}r^2 + C_1 \quad \rightarrow \quad T(r) = -\frac{\dot{q}}{4k}r^2 + C_1 \ln r + C_2$$

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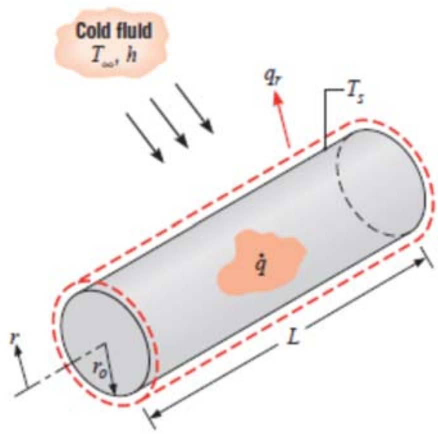
Cylindrical Systems

- Boundary conditions

$$\left.\frac{dT}{dr}\right|_{r=0} = 0 \quad \text{and} \quad T(r_0) = T_s$$

$$C_1 = 0. \quad \text{and} \quad C_2 = T_s + \frac{\dot{q}}{4k} r_o^2$$



The temperature distribution

$$T(r) = \frac{\dot{q} r_o^2}{4k}\left(1 - \frac{r^2}{r_o^2}\right) + T_s$$

$$\text{at } r = 0 \quad T = T_o \quad \rightarrow \quad T_0 = \frac{\dot{q} R^2}{4k} + T_s$$

➢ The temperature distribution in non-dimensional form $\dfrac{T(r) - T_s}{T_o - T_s} = 1 - \left(\dfrac{r}{r_o}\right)^2$

Cylindrical Systems

Surface Temperature

Overall energy balance:

$$\dot{E}_g = \dot{E}_{out}$$

$$\dot{q}(\pi r_o^2 L) = h(2\pi r_o L)(T_s - T_\infty)$$

$$T_s = T_\infty + \frac{\dot{q} r_o}{2h}$$

surface energy balance:

$$-k\left.\frac{dT}{dr}\right|_{r=r_o} = h(T_s - T_\infty)$$

➢ A summary of temperature distributions is provided in Appendix C for plane, cylindrical and spherical walls, as well as for solid cylinders and spheres. Note how boundary conditions are specified and how they are used to obtain surface temperatures.

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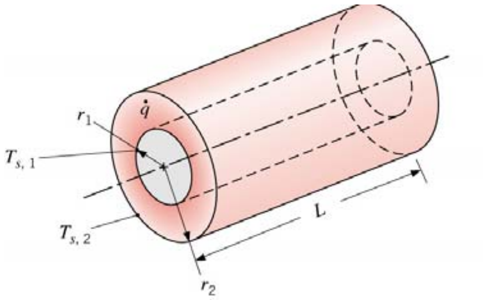
Cylindrical Systems

- For a hollow cylinder with uniformly distributed heat sources the appropriate boundary conditions would be

$$T = T_i \quad \text{at } r = r_i \text{ (inside surface)}$$
$$T = T_o \quad \text{at } r = r_o \text{ (outside surface)}$$



The general solution is still

$$T = -\frac{\dot{q}r^2}{4k} + C_1 \ln r + C_2$$

$$\Rightarrow \quad T - T_o = \frac{\dot{q}}{4k}(r_o^2 - r^2) + C_1 \ln \frac{r}{r_o}$$

where $$C_1 = \frac{T_i - T_o + \dot{q}(r_i^2 - r_o^2)/4k}{\ln (r_i/r_o)}$$

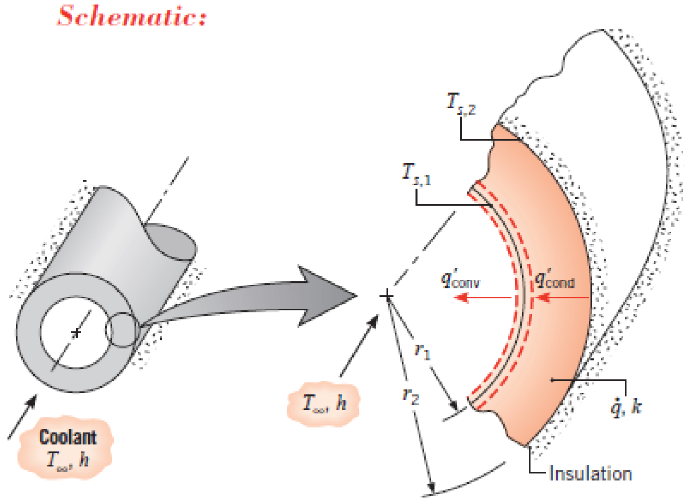
Example

Consider a long solid tube, insulated at the outer radius r_2 and cooled at the inner radius r_1, with uniform heat generation $\dot{q}$ (W/m^3) within the solid.

1. Obtain the general solution for the temperature distribution in the tube.
2. In a practical application a limit would be placed on the maximum temperature that is permissible at the insulated surface ($r = r_2$). Specifying this limit as $T_{s,2}$, identify appropriate boundary conditions that could be used to determine the arbitrary constants appearing in the general solution. Determine these constants and the corresponding form of the temperature distribution.
3. Determine the heat removal rate per unit length of tube.
4. If the coolant is available at a temperature T_∞, obtain an expression for the convection coefficient that would have to be maintained at the inner surface to allow for operation at prescribed values of $T_{s,2}$ and $\dot{q}$.

Example cont.

Schematic:



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Example cont.

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Example cont.

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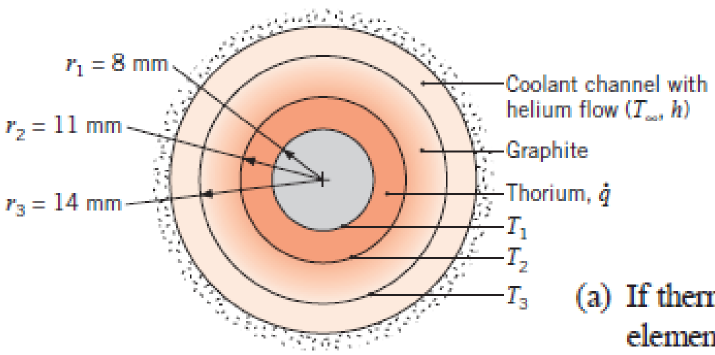
Example

3.100 A high-temperature, gas-cooled nuclear reactor consists of a composite cylindrical wall for which a thorium fuel element ($k \approx 57$ W/m·K) is encased in graphite ($k \approx 3$ W/m·K) and gaseous helium flows through an annular coolant channel. Consider conditions for which the helium temperature is $T_\infty = 600$ K and the convection coefficient at the outer surface of the graphite is $h = 2000$ W/m^2·K.



(a) If thermal energy is uniformly generated in the fuel element at a rate $\dot{q} = 10^8$ W/m^3, what are the temperatures T_1 and T_2 at the inner and outer surfaces, respectively, of the fuel element?

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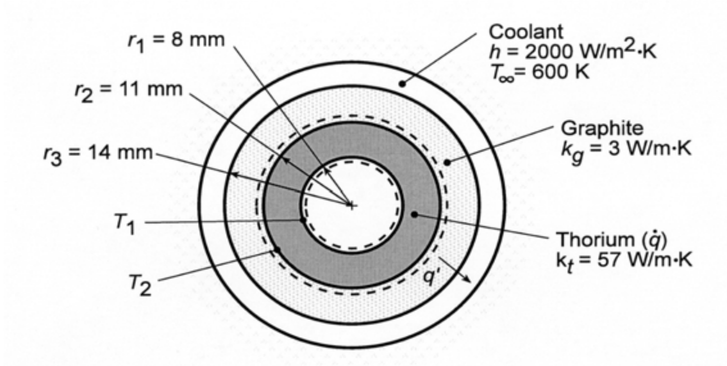
Example cont.

Schematic:

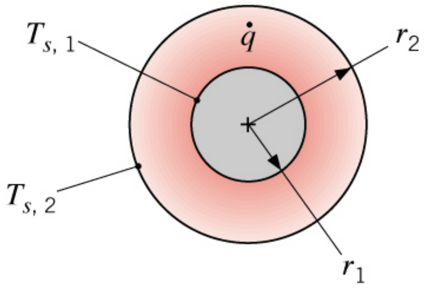


Example cont.

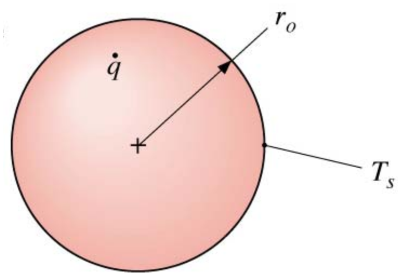
Example cont.

Spheres

Spherical Wall (Shell)



Solid Sphere



- Heat Equations: $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)+\frac{\dot{g}}{k}=0$$

Spheres

Temperature Distribution

$$kr^2 \frac{dT}{dr} = -\frac{\dot{q}\, r^3}{3} + C_1 \quad \Rightarrow \quad T = -\frac{\dot{q}\, r^2}{6k} - \frac{C_1}{r} + C_2$$

- B.C 1:

$$\frac{dT}{dr}\Big|_{r=0} = 0 \rightarrow C_1 = 0$$

- B.C 2:

$$T(r_o) = T_s \rightarrow C_2 = T_s + \frac{\dot{q}\, r_o^2}{6k}$$

$$\Rightarrow \quad T(r) = \frac{\dot{q}\, r_o^2}{6k}\left(1 - \frac{r^2}{r_o^2}\right) + T_s$$

Spheres

Surface Temperature

Overall energy balance:

$$-\dot{E}_{out} + \dot{E}_g = 0 \qquad \rightarrow T_s = T_\infty + \frac{\dot{q}\, r_o}{3h}$$

Or from a surface energy balance:

$$\dot{E}_{in} - \dot{E}_{out} = 0 \qquad \rightarrow q_{cond}(r_o) = q_{conv} \qquad \rightarrow T_s = T_\infty + \frac{\dot{q}\, r_o}{3h}$$

Example

3.72 A composite spherical shell of inner radius $r_1 = 0.25$ m is constructed from lead of outer radius $r_2 = 0.30$ m and AISI 302 stainless steel of outer radius $r_3 = 0.31$ m. The cavity is filled with radioactive wastes that generate heat at a rate of $\dot{q} = 5 \times 10^5$ W/m^3. It is proposed to submerge the container in oceanic waters that are at a temperature of $T_\infty = 10°$C and provide a uniform convection coefficient of $h = 500$ W/m$^2 \cdot$K at the outer surface of the container. Are there any problems associated with this proposal?

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Example cont.

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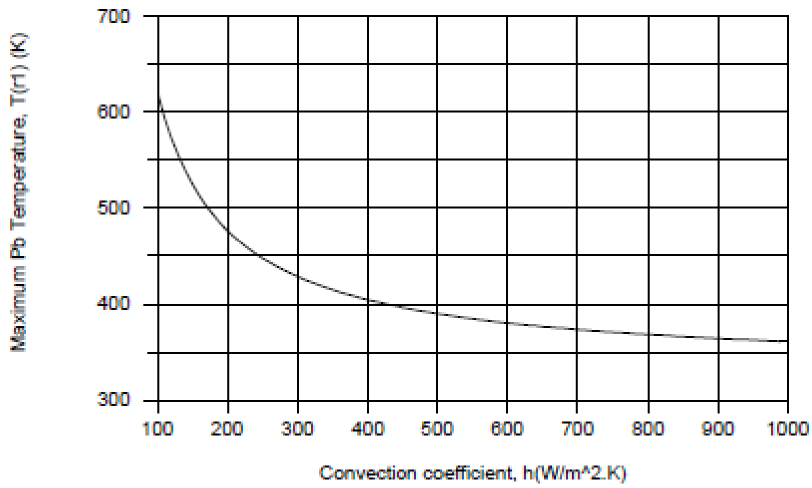
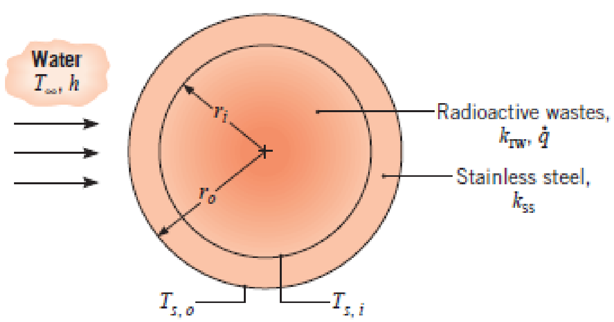
Example cont.



Example

3.104 Radioactive wastes (k_{rw} = 20 W/m·K) are stored in a spherical, stainless steel (k_{ss} = 15 W/m·K) container of inner and outer radii equal to r_i = 0.5 m and r_o = 0.6 m. Heat is generated volumetrically within the wastes at a uniform rate of $\dot{q} = 10^5$ W/m^3, and the outer surface of the container is exposed to a water flow for which h = 1000 W/m^2·K and T_∞ = 25°C.



(a) Evaluate the steady-state outer surface temperature, $T_{s,o}$.

(b) Evaluate the steady-state inner surface temperature, $T_{s,i}$.

(c) Obtain an expression for the temperature distribution, $T(r)$, in the radioactive wastes. Express your result in terms of r_i, $T_{s,i}$, k_{rw}, and $\dot{q}$. Evaluate the temperature at $r = 0$.

Example cont.

Example cont.

Example cont.

TABLE 3.3 One-dimensional, steady-state solutions to the heat equation with no generation

	Plane Wall	Cylindrical Wall[a]	Spherical Wall[a]
Heat equation	$\frac{d^2T}{dx^2} = 0$	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$	$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$
Temperature distribution	$T_{s,1} - \Delta T\frac{x}{L}$	$T_{s,2} + \Delta T\frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1} - \Delta T\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$
Heat flux (q'')	$k\frac{\Delta T}{L}$	$\frac{k\Delta T}{r\ln(r_2/r_1)}$	$\frac{k\Delta T}{r^2[(1/r_1)-(1/r_2)]}$
Heat rate (q)	$kA\frac{\Delta T}{L}$	$\frac{2\pi Lk\Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k\Delta T}{(1/r_1)-(1/r_2)}$
Thermal resistance ($R_{t,cond}$)	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1)-(1/r_2)}{4\pi k}$

[a]The critical radius of insulation is $r_{cr} = k/h$ for the cylinder and $r_{cr} = 2k/h$ for the sphere.