# One-Dimensional, Steady-State Conduction without Thermal Energy Generation

## Chapter Three

## Sections 3.1 through 3.4

# Methodology of a Conduction Analysis

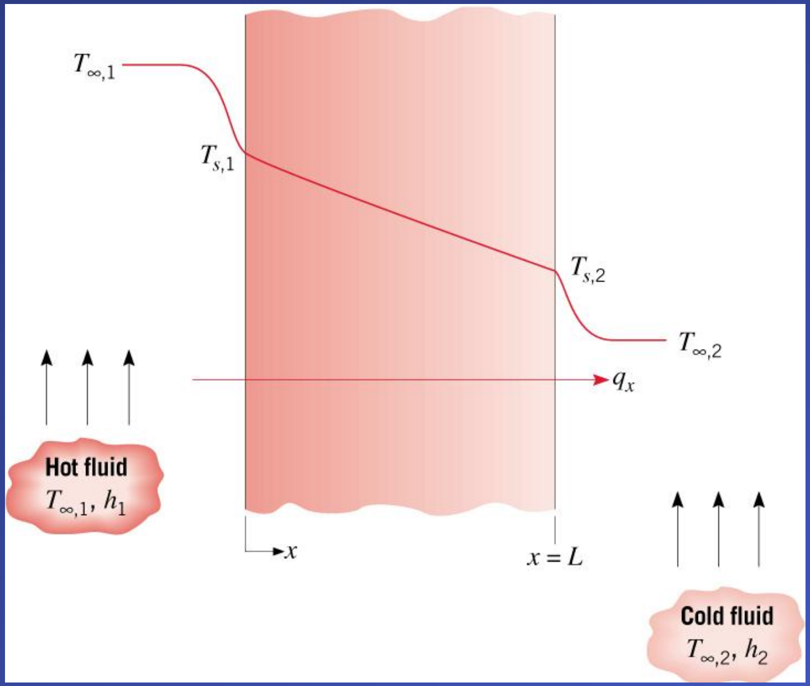
- Specify appropriate form of the heat equation.
- Solve for the temperature distribution.
- Apply Fourier's law to determine the heat flux.

Simplest Case: One-Dimensional, Steady-State Conduction with No Thermal Energy Generation.

- Common Geometries:
  - The Plane Wall: Described in rectangular ($x$) coordinate. Area perpendicular to direction of heat transfer is constant (independent of $x$).
  - The Tube Wall: Radial conduction through tube wall.
  - The Spherical Shell: Radial conduction through shell wall.

# The Plane Wall

- Consider a plane wall between two fluids of different temperature:




- Heat Equation:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0 \qquad (3.1)$$

- Implications:

  Heat flux $\left(q''_x\right)$ is independent of $x$.

  Heat rate $\left(q_x\right)$ is independent of $x$.

- Boundary Conditions: $T(0)=T_{s,1}, \quad T(L)=T_{s,2}$

- Temperature Distribution for Constant $k$:

$$T(x)=T_{s,1}+\left(T_{s,2}-T_{s,1}\right)\frac{x}{L} \qquad (3.3)$$

- Heat Flux and Heat Rate:

$$q''_x = -k\frac{dT}{dx} = \frac{k}{L}\left(T_{s,1} - T_{s,2}\right) \qquad (3.5)$$

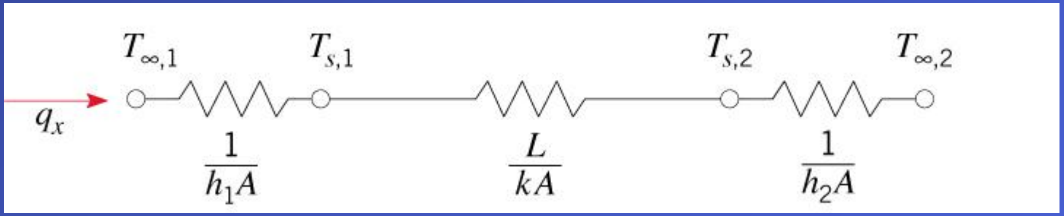
$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}\left(T_{s,1} - T_{s,2}\right) \qquad (3.4)$$

- Thermal Resistances $\left(R_t = \frac{\Delta T}{q}\right)$ and Thermal Circuits:

Conduction in a plane wall: $$R_{t,\text{cond}} = \frac{L}{kA} \qquad (3.6)$$

Convection: $$R_{t,\text{conv}} = \frac{1}{hA} \qquad (3.9)$$

Thermal circuit for plane wall with adjoining fluids:




$$R_{\text{tot}} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A} \qquad (3.12)$$

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{\text{tot}}} \qquad (3.11)$$

- Thermal Resistance for Unit Surface Area:

$$R''_{t,\text{cond}} = \frac{L}{k} \qquad R''_{t,\text{conv}} = \frac{1}{h}$$

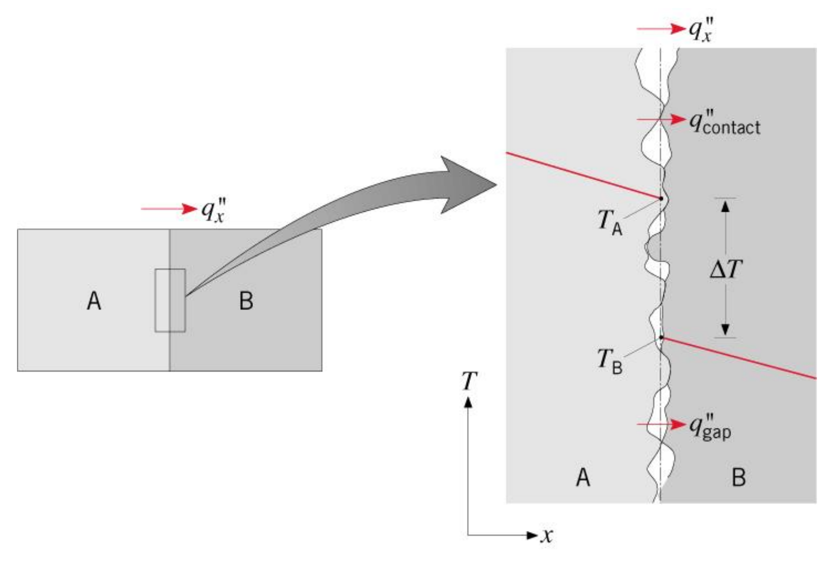
Units: $R_t \leftrightarrow \text{K/W} \qquad R''_t \leftrightarrow \text{m}^2 \cdot \text{K/W}$

- Radiation Resistance:

$$R_{t,\text{rad}} = \frac{1}{h_r A} \qquad R''_{t,\text{rad}} = \frac{1}{h_r}$$

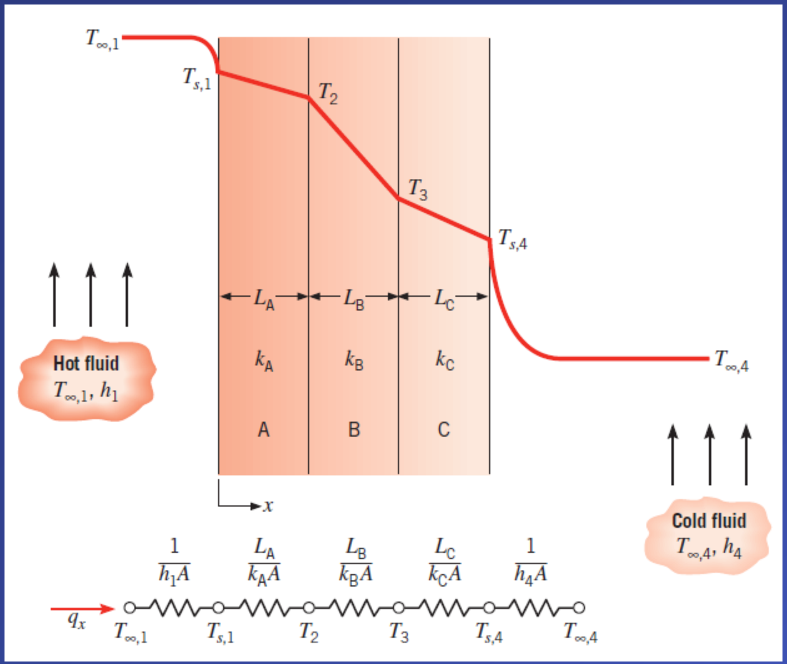
$$h_r = \varepsilon\sigma\left(T_s + T_{\text{sur}}\right)\left(T_s^2 + T_{\text{sur}}^2\right) \qquad (1.9)$$

- Contact Resistance:




$$R''_{t,c} = \frac{T_A - T_B}{q''_x} \qquad R_{t,c} = \frac{R''_{t,c}}{A_c}$$

Values depend on: Materials A and B, surface finishes, interstitial conditions, and contact pressure (Tables 3.1 and 3.2)




- **Composite Wall with Negligible Contact Resistance:**

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\sum R_t} \qquad (3.14)$$

For the temperature distribution shown, $k_A > k_B < k_C$.

$$\sum R_t = R_{tot} = \frac{1}{A}\left[\frac{1}{h_1} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_4}\right] = \frac{R''_{tot}}{A}$$

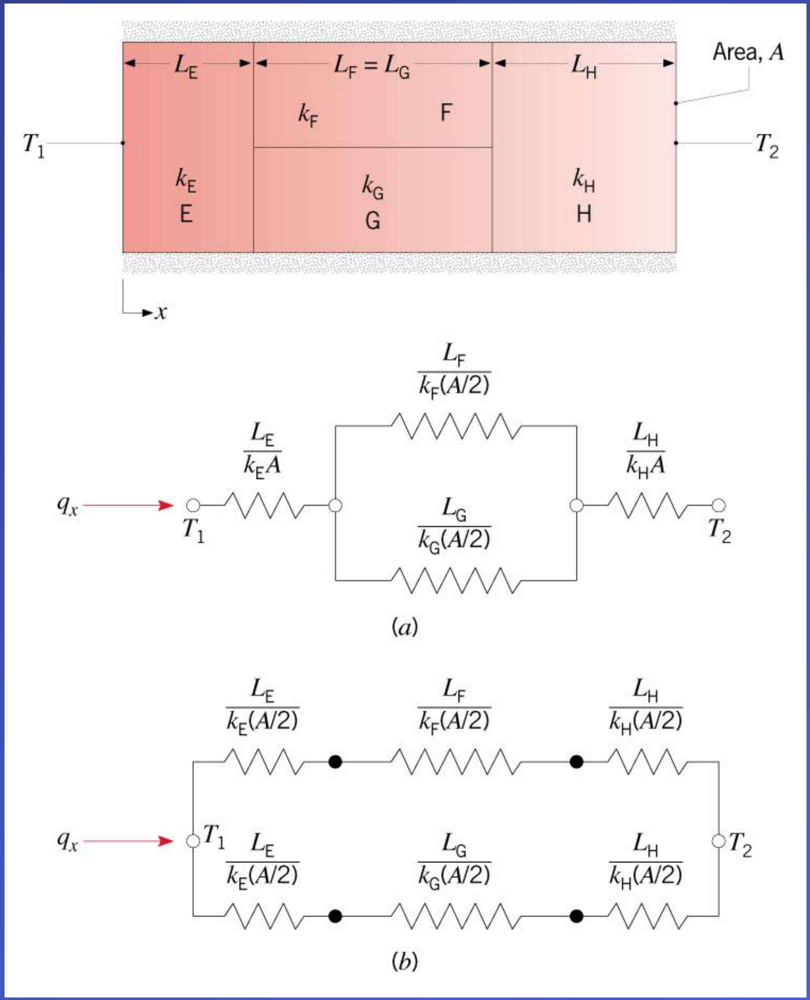
- **Overall Heat Transfer Coefficient ($U$) :**

A modified form of Newton's law of cooling to encompass multiple resistances to heat transfer.

$$q_x = UA\Delta T_{overall} \qquad (3.17)$$

$$R_{tot} = \frac{1}{UA} \qquad (3.19)$$

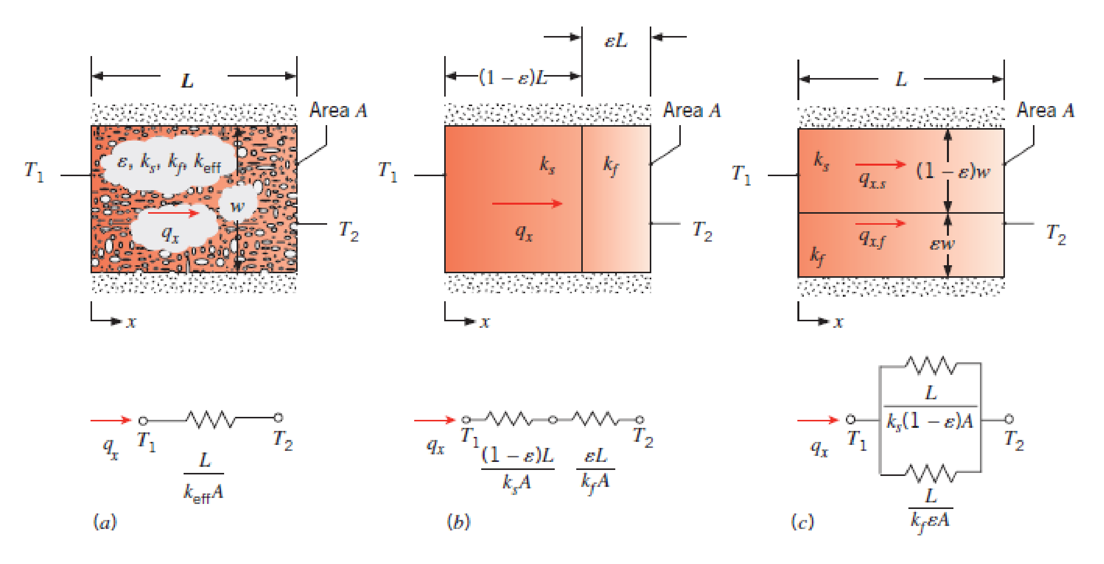
- Series – Parallel Composite Wall:




- Note departure from one-dimensional conditions for $k_F \neq k_G$ .

- Circuits based on assumption of isothermal surfaces normal to $x$ direction or adiabatic surfaces parallel to $x$ direction provide approximations for $q_x$ .

- Porous Media




- Saturated media consist of a solid phase and a single fluid phase.
- Unsaturated media consist of solid, liquid, and gas phases.

- The effective thermal conductivity of a saturated medium depends on the solid ($s$) material, its porosity ε, its morphology, as well as the interstitial fluid ($f$) (Fig.a).

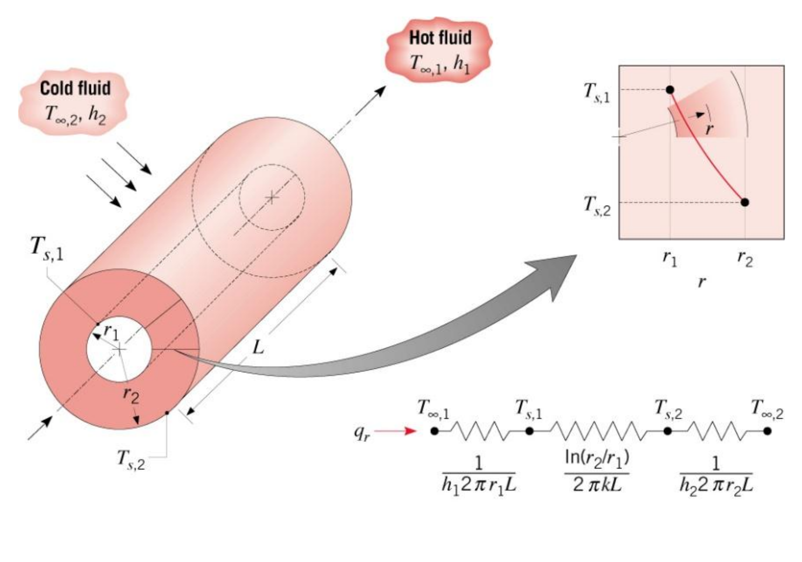
$$q_x = \frac{k_{\text{eff}} A}{L}(T_1 - T_2) \qquad (3.21)$$

- The value of $k_{\text{eff}}$ may be *bracketed* by describing the medium with a series resistance analysis (Fig. b) and a parallel resistance analysis (Fig.c).
- The value of $k_{\text{eff}}$ may be *estimated* by

$$k_{\text{eff}} = \left[\frac{k_f + 2k_s - 2\varepsilon(k_s - k_f)}{k_f + 2k_s + \varepsilon(k_s - k_f)}\right] k_s \qquad (3.25)$$

$$\varepsilon \leq 0.25$$

# The Tube Wall




- Heat Equation:

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right)=0 \qquad (3.28)$$

What does the form of the heat equation tell us about the variation of $q_r$ with $r$ in the wall?

Is the foregoing conclusion consistent with the energy conservation requirement?

How does $q''_r$ vary with $r$?

- Temperature Distribution for Constant $k$:

$$T(r)=\frac{T_{s,1}-T_{s,2}}{\ln(r_1/r_2)}\ln\left(\frac{r}{r_2}\right)+T_{s,2} \qquad (3.31)$$

- Heat Flux and Heat Rate:

$$q''_r = -k\frac{dT}{dr} = \frac{k}{r\ln(r_2/r_1)}\left(T_{s,1} - T_{s,2}\right) \qquad [\text{W/m}^2]$$

$$q'_r = 2\pi r q''_r = \frac{2\pi k}{\ln(r_2/r_1)}\left(T_{s,1} - T_{s,2}\right) \qquad [\text{W/m}]$$

$$q_r = 2\pi r L q''_r = \frac{2\pi L k}{\ln(r_2/r_1)}\left(T_{s,1} - T_{s,2}\right) \qquad [\text{W}] \qquad (3.32)$$

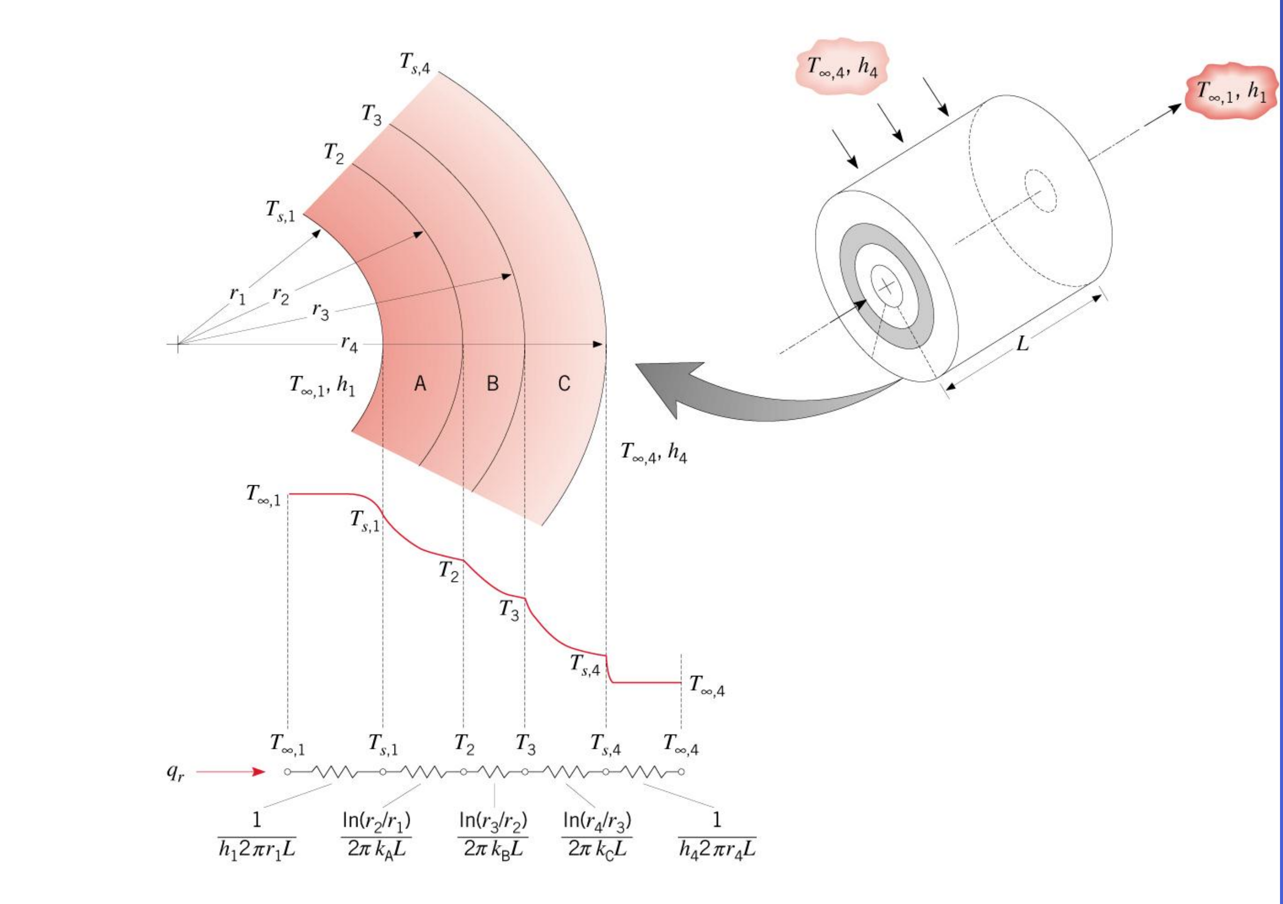
- Conduction Resistance:

$$R_{t,\text{cond}} = \frac{\ln(r_2/r_1)}{2\pi L k} \qquad [\text{K/W}] \qquad (3.33)$$

$$R'_{t,\text{cond}} = \frac{\ln(r_2/r_1)}{2\pi k} \qquad [\text{m}\cdot\text{K/W}]$$

Why doesn't a surface area appear in the expressions for the thermal resistance?

- Composite Wall with Negligible Contact Resistance




$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{tot}}$$

$$= UA\left(T_{\infty,1} - T_{\infty,4}\right) \quad (3.35)$$

For the temperature distribution shown, $k_A > k_B > k_C$.

Note that

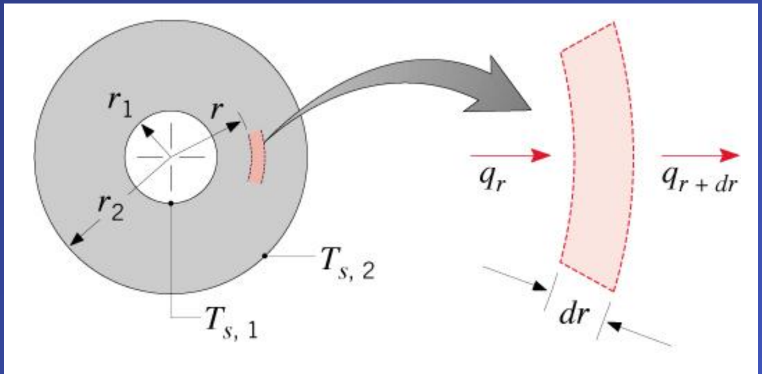
$$UA = R_{tot}^{-1}$$

is a constant independent of radius,

but $U$ itself is tied to specification of an interface.

$$U_i = \left(A_i R_{tot}\right)^{-1} \quad (3.37)$$

# Spherical Shell




- Heat Equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)=0$$

What does the form of the heat equation tell us about the variation of $q_r$ with $r$? Is this result consistent with conservation of energy?

How does $q''_r$ vary with $r$?

- Temperature Distribution for Constant $k$:

$$T(r)=T_{s,1}-\left(T_{s,1}-T_{s,2}\right)\frac{1-\left(r_1/r\right)}{1-\left(r_1/r_2\right)}$$

- Heat flux, Heat Rate and Thermal Resistance:

$$q''_r = -k\frac{dT}{dr} = \frac{k}{r^2\left[(1/r_1)-(1/r_2)\right]}\left(T_{s,1}-T_{s,2}\right)$$

$$q_r = 4\pi r^2 q''_r = \frac{4\pi k}{(1/r_1)-(1/r_2)}\left(T_{s,1}-T_{s,2}\right) \qquad (3.40)$$

$$R_{t,\text{cond}} = \frac{(1/r_1)-(1/r_2)}{4\pi k} \qquad (3.41)$$

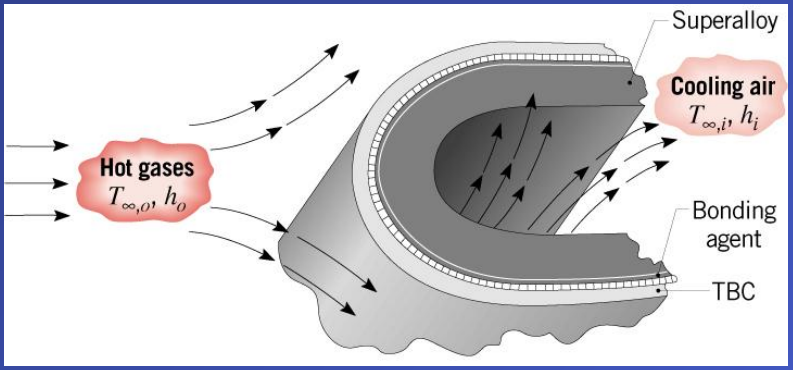
- Composite Shell:

$$q_r = \frac{\Delta T_{\text{overall}}}{R_{\text{tot}}} = UA\Delta T_{\text{overall}}$$

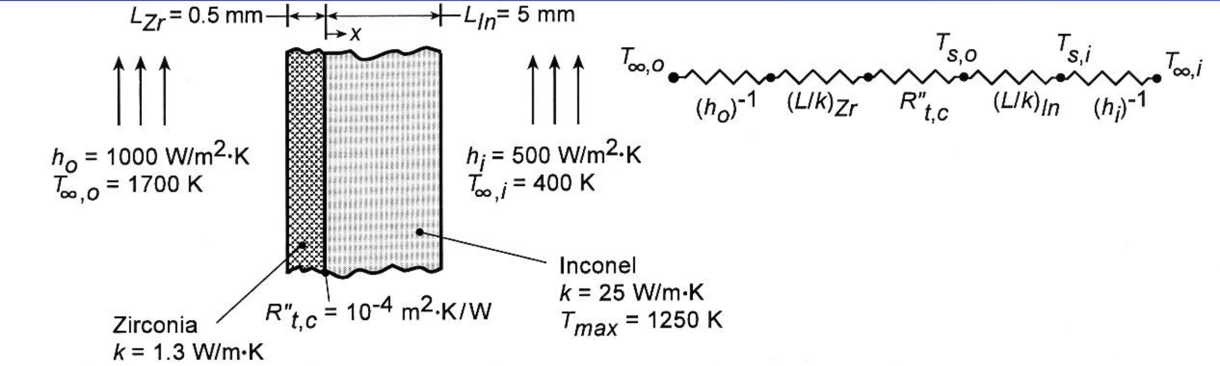
$$UA = R_{\text{tot}}^{-1} \leftrightarrow \text{Constant}$$

$$U_i = \left(A_i R_{\text{tot}}\right)^{-1} \leftrightarrow \text{Depends on } A_i$$

# Problem 3.30: Assessment of thermal barrier coating (TBC) for protection of turbine blades. Determine maximum blade temperature with and without TBC.




## Schematic:




**ASSUMPTIONS:** (1) One-dimensional, steady-state conduction in a composite plane wall, (2) Constant properties, (3) Negligible radiation.

**ANALYSIS:** For a unit area, the total thermal resistance with the TBC is

$$R''_{tot,w} = h_o^{-1} + (L/k)_{Zr} + R''_{t,c} + (L/k)_{In} + h_i^{-1}$$

$$R''_{tot,w} = \left(10^{-3} + 3.85\times10^{-4} + 10^{-4} + 2\times10^{-4} + 2\times10^{-3}\right)\mathrm{m^2\cdot K/W} = 3.69\times10^{-3}\ \mathrm{m^2\cdot K/W}$$

With a heat flux of

$$q''_w = \frac{T_{\infty,o} - T_{\infty,i}}{R''_{tot,w}} = \frac{1300\,\mathrm{K}}{3.69\times10^{-3}\,\mathrm{m^2\cdot K/W}} = 3.52\times10^{5}\ \mathrm{W/m^2}$$

the inner and outer surface temperatures of the Inconel are

$$T_{s,i(w)} = T_{\infty,i} + (q''_w/h_i) = 400\,\mathrm{K} + \left(\frac{3.52\times10^5\,\mathrm{W/m^2}}{500\,\mathrm{W/m^2\cdot K/W}}\right) = 1104\,\mathrm{K}$$

$$T_{s,o(w)} = T_{\infty,i} + \left[(1/h_i) + (L/k)_{In}\right]q''_w = 400\,\mathrm{K} + \left(2\times10^{-3} + 2\times10^{-4}\right)\mathrm{m^2\cdot K/W}\left(3.52\times10^5\,\mathrm{W/m^2}\right) = 1174\,\mathrm{K} \quad <$$

Without the TBC,

$$R''_{\text{tot,wo}} = h_o^{-1} + \left(L/k\right)_{\text{In}} + h_i^{-1} = 3.20\times10^{-3}\,\text{m}^2\cdot\text{K/W}$$

$$q''_{\text{wo}} = \left(T_{\infty,o} - T_{\infty,i}\right)/R''_{\text{tot,wo}} = 4.06\times10^{5}\ \text{W/m}^2$$

The inner and outer surface temperatures of the Inconel are then

$$T_{s,i(\text{wo})} = T_{\infty,i} + \left(q''_{wo}/h_i\right) = 1212\,\text{K}$$

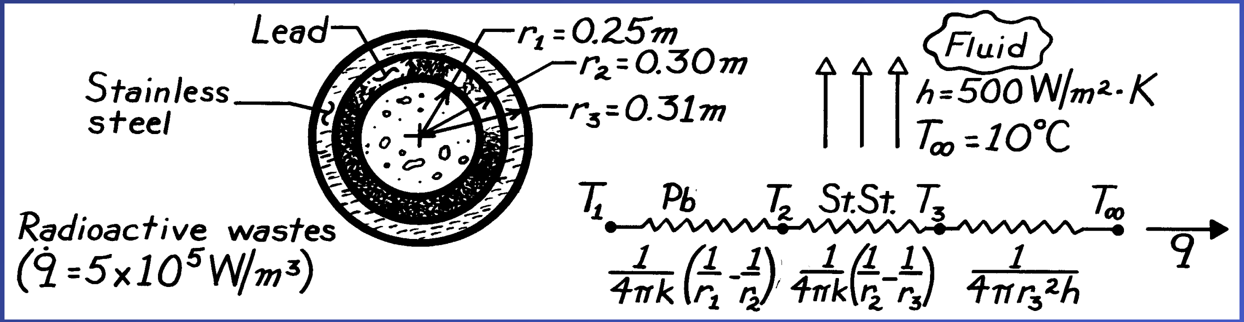
$$T_{s,o(\text{wo})} = T_{\infty,i} + \left[\left(1/h_i\right) + \left(L/k\right)_{\text{In}}\right] q''_{wo} = 1293\,\text{K}$$ <

Use of the TBC facilitates operation of the Inconel below $T_{\text{max}} = 1250$ K.

**COMMENTS:** Since the durability of the TBC decreases with increasing temperature, which increases with increasing thickness, limits to its thickness are associated with reliability considerations.

# Problem 3.72: Suitability of a composite spherical shell for storing radioactive wastes in oceanic waters.

**SCHEMATIC:**




**ASSUMPTIONS:** (1) One-dimensional conduction, (2) Steady-state conditions, (3) Constant properties at 300K, (4) Negligible contact resistance.

**PROPERTIES**: Table A-1, Lead: $k$ = 35.3 W/m·K, MP = 601 K; St.St.: $k$ = 15.1 W/m·K.

**ANALYSIS:** From the thermal circuit, it follows that

$$q = \frac{T_1 - T_\infty}{R_{tot}} = \dot{q}\left[\frac{4}{3}\pi\, r_1^3\right]$$

The thermal resistances are:

$$R_{Pb} = \left[1/\left(4\pi \times 35.3\ \text{W/m}\cdot\text{K}\right)\right]\left[\frac{1}{0.25\text{m}} - \frac{1}{0.30\text{m}}\right] = 0.00150\ \text{K/W}$$

$$R_{St.St.} = \left[1/\left(4\pi \times 15.1\ \text{W/m}\cdot\text{K}\right)\right]\left[\frac{1}{0.30\text{m}} - \frac{1}{0.31\text{m}}\right] = 0.000567\ \text{K/W}$$

$$R_{conv} = \left[1/\left(4\pi \times 0.31^2\,\text{m}^2 \times 500\ \text{W/m}^2\cdot\text{K}\right)\right] = 0.00166\ \text{K/W}$$

$$R_{tot} = 0.00372\ \text{K/W}$$

The heat rate is then

$$q = 5\times10^5\ \text{W/m}^3\left(4\pi/3\right)\left(0.25\text{m}\right)^3 = 32,725\ \text{W}$$

and the inner surface temperature is

$$T_1 = T_\infty + R_{tot}q = 283\ \text{K} + 0.00372\ \text{K/W}\left(32,725\ \text{W}\right)$$

$$= 405\ \text{K} < \text{MP} = 601\ \text{K}$$

<

Hence, from the thermal standpoint, the proposal is adequate.

**COMMENTS:** In fabrication, attention should be given to maintaining a good thermal contact. A protective outer coating should be applied to prevent long term corrosion of the stainless steel.

# One-Dimensional, Steady-State Conduction with Thermal Energy Generation

Chapter Three

Section 3.5, Appendix C

# Implications of Energy Generation

- Involves a local (volumetric) source of thermal energy due to conversion from another form of energy in a conducting medium.

- The source may be uniformly distributed, as in the conversion from electrical to thermal energy (Ohmic heating):

$$\dot{q} = \frac{\dot{E}_g}{\forall} = \frac{I^2 R_e}{\forall} \qquad (3.43)$$

or it may be non-uniformly distributed, as in the absorption of radiation passing through a semi-transparent medium.

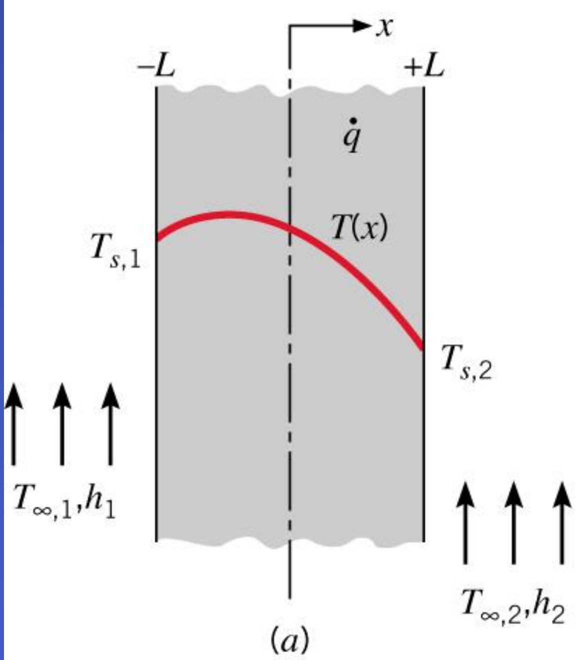
For a plane wall,

$$\dot{q} \propto e^{-\alpha x}$$

- Generation affects the temperature distribution in the medium and causes the heat rate to vary with location, thereby precluding inclusion of the medium in a thermal circuit.

# The Plane Wall

- Consider one-dimensional, steady-state conduction in a plane wall of constant $k$, uniform generation, and asymmetric surface conditions:




- Heat Equation:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+\dot{q}=0 \rightarrow \frac{d^2T}{dx^2}+\frac{\dot{q}}{k}=0 \qquad (3.44)$$

Is the heat flux $q''$ independent of $x$?

- General Solution:

$$T(x)=-(\dot{q}/2k)x^2+C_1x+C_2 \qquad (3.45)$$

What is the form of the temperature distribution for

$\dot{q}=0?$ $\quad \dot{q}>0?$ $\quad \dot{q}<0?$

How does the temperature distribution change with increasing $\dot{q}$?

# Plane wall (cont.)

Symmetric Surface Conditions or One Surface Insulated:

- What is the temperature gradient at the centerline or the insulated surface?

- Why does the magnitude of the temperature gradient increase with increasing *x*?

- Temperature Distribution:

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s \qquad (3.47)$$

- How do we determine $T_s$?

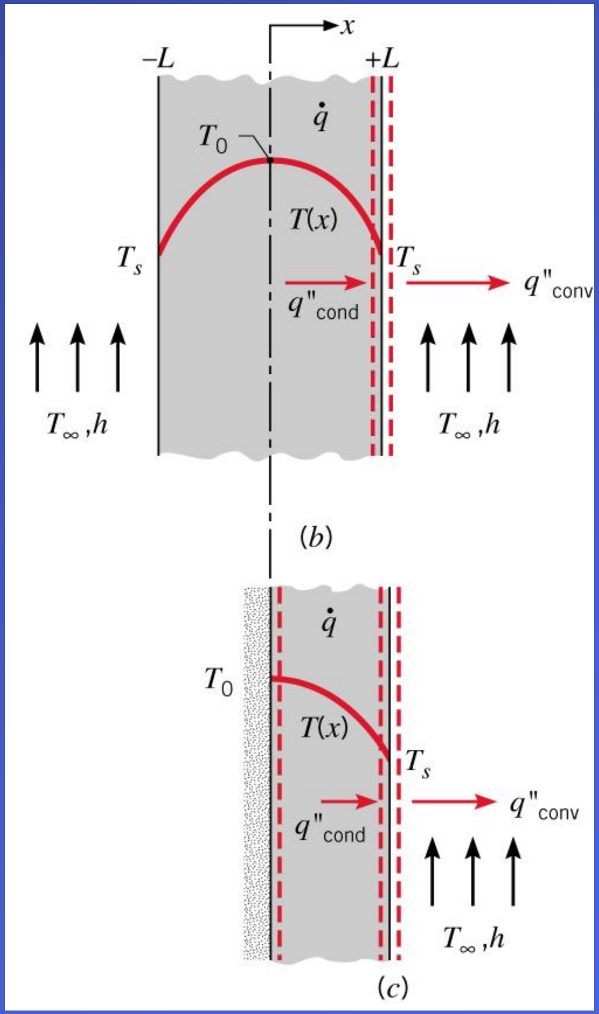
  Overall energy balance on the wall →

$$-\dot{E}_{out} + \dot{E}_g = 0$$

$$-hA_s\left(T_s - T_\infty\right) + \dot{q}A_sL = 0$$

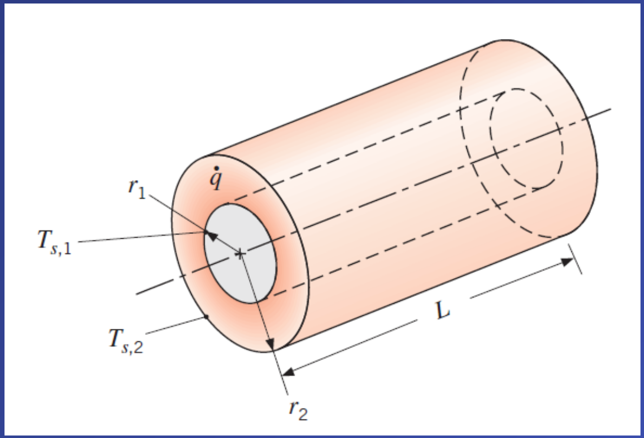
$$T_s = T_\infty + \frac{\dot{q}L}{h} \qquad (3.51)$$

- How do we determine the heat rate at $x = L$?

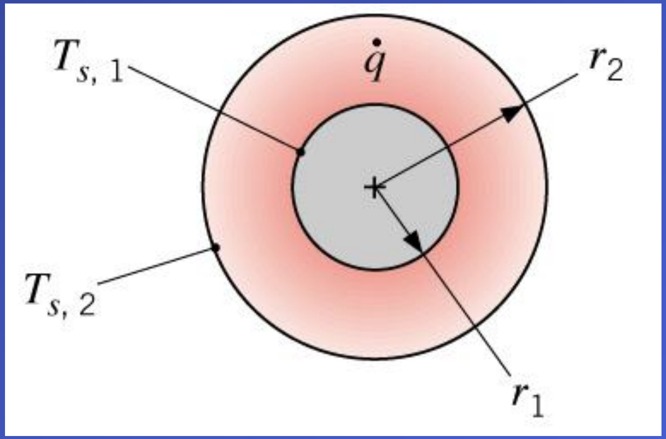


# Radial Systems

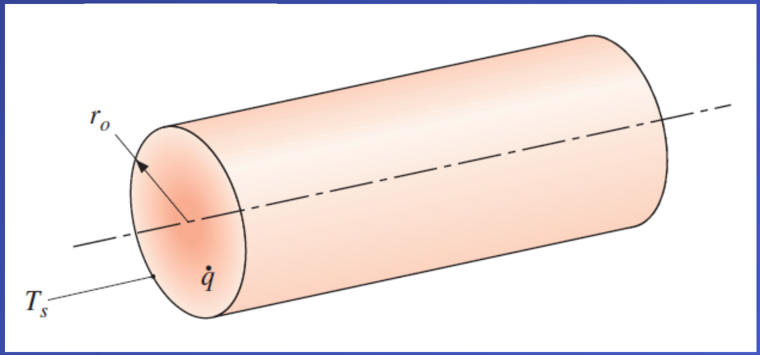
Cylindrical (Tube) Wall



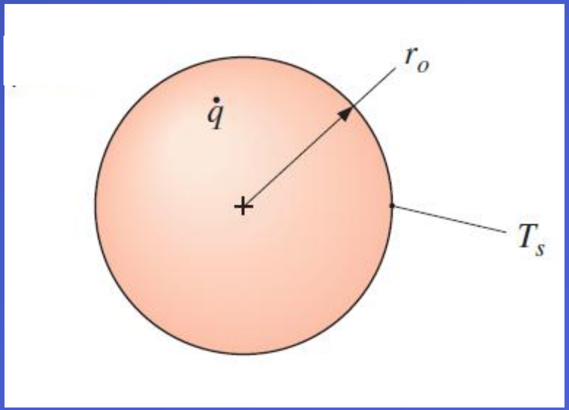

Spherical Wall (Shell)




Solid Cylinder (Circular Rod)




Solid Sphere




- Heat Equations:

Cylindrical

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right)+\dot{q}=0$$

Spherical

$$\frac{1}{r^2}\frac{d}{dr}\left(kr^2\frac{dT}{dr}\right)+\dot{q}=0$$

- Solution for Uniform Generation in a Solid Sphere of Constant $k$ with Convection Cooling:

Temperature Distribution

$$kr^2 \frac{dT}{dr} = -\frac{\dot{q}\, r^3}{3} + C_1$$

$$T = -\frac{\dot{q}\, r^2}{6k} - \frac{C_1}{r} + C_2$$

$$\frac{dT}{dr}\Big|_{r=0} = 0 \rightarrow C_1 = 0$$

$$T(r_o) = T_s \rightarrow C_2 = T_s + \frac{\dot{q}\, r_o^2}{6k}$$

$$T(r) = \frac{\dot{q}\, r_o^2}{6k}\left(1 - \frac{r^2}{r_o^2}\right) + T_s$$

Surface Temperature

Overall energy balance:

$$-\dot{E}_{\text{out}} + \dot{E}_g = 0 \rightarrow T_s = T_\infty + \frac{\dot{q}\, r_o}{3h}$$

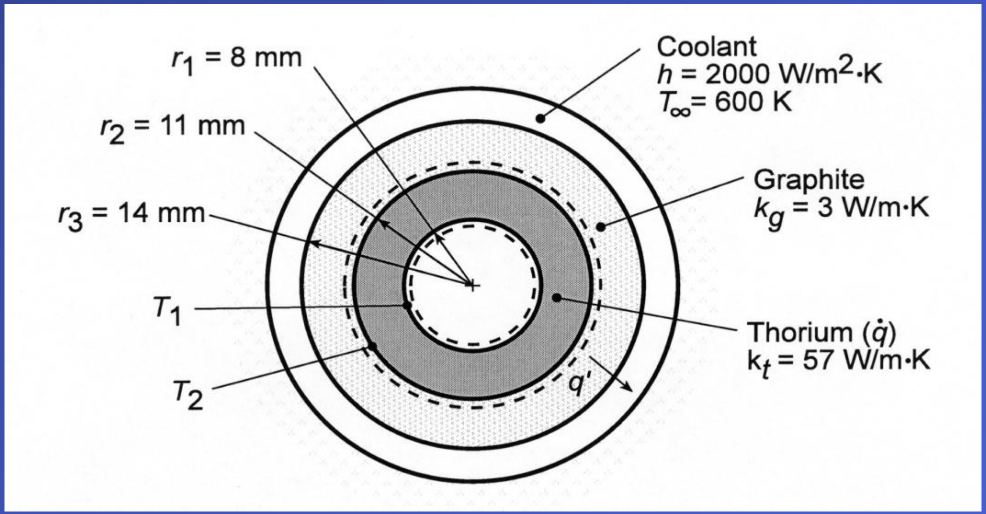
Or from a surface energy balance:

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = 0 \rightarrow q_{\text{cond}}(r_o) = q_{\text{conv}} \rightarrow T_s = T_\infty + \frac{\dot{q}\, r_o}{3h}$$

- A summary of temperature distributions is provided in Appendix C for plane, cylindrical and spherical walls, as well as for solid cylinders and spheres. Note how boundary conditions are specified and how they are used to obtain surface temperatures.

Problem 3.100 Thermal conditions in a gas-cooled nuclear reactor with a tubular thorium fuel rod and a concentric graphite sheath: (a) Assessment of thermal integrity for a generation rate of $\dot{q} = 10^8\ \text{W/m}^3$. (b) Evaluation of temperature distributions in the thorium and graphite for generation rates in the range $10^8 \le \dot{q} \le 5\text{x}10^8$

**Schematic:**




**Assumptions:** (1) Steady-state conditions, (2) One-dimensional conduction, (3) Constant properties, (4) Negligible contact resistance, (5) Negligible radiation.

**Properties:** Table A.1, Thorium: $T_{mp} \approx 2000\text{K}$; Table A.2, Graphite: $T_{mp} \approx 2300\text{K}$.

# Problem: Nuclear fuel rod (cont.)

**Analysis**: (a) The outer surface temperature of the fuel, *T2* , may be determined from the rate equation

$$q' = \frac{T_2 - T_\infty}{R'_{\text{tot}}}$$

where $$R'_{\text{tot}} = \frac{\ln(r_3 / r_2)}{2\pi k_g} + \frac{1}{2\pi r_3 h} = 0.0185 \text{ m}\cdot\text{K/W}$$

The heat rate may be determined by applying an energy balance to a control surface about the fuel element,

$$\dot{E}_{\text{out}} = \dot{E}_g$$

or, per unit length,

$$\dot{E}'_{\text{out}} = \dot{E}'_g$$

Since the interior surface of the element is essentially adiabatic, it follows that

$$q' = \dot{q}\pi\left(r_2^2 - r_1^2\right) = 17{,}907 \text{ W/m}$$

Hence,

$$T_2 = q'R'_{\text{tot}} + T_\infty = 17{,}907 \text{ W/m}\left(0.0185 \text{ m}\cdot\text{K/W}\right) + 600\text{K} = 931\text{K}$$

With zero heat flux at the inner surface of the fuel element, Eq. C.14 yields

$$T_1 = T_2 + \frac{\dot{q} r_2^2}{4k_t}\left(1 - \frac{r_1^2}{r_2^2}\right) - \frac{\dot{q} r_1^2}{2k_t}\ln\left(\frac{r_2}{r_1}\right) = 931\text{K} + 25\text{K} - 18\text{K} = 938\text{K}$$ <

Since *T1* and *T2* are well below the melting points of thorium and graphite, the prescribed operating condition is acceptable.

(b) The solution for the temperature distribution in a cylindrical wall with generation is

$$T_t(r) = T_2 + \frac{\dot{q}\, r_2^2}{4k_t}\left(1 - \frac{r^2}{r_2^2}\right) - \left[\frac{\dot{q}\, r_2^2}{4k_t}\left(1 - \frac{r_1^2}{r_2^2}\right) + \left(T_2 - T_1\right)\right]\frac{\ln(r_2/r)}{\ln(r_2/r_1)} \qquad \text{(C.2)}$$

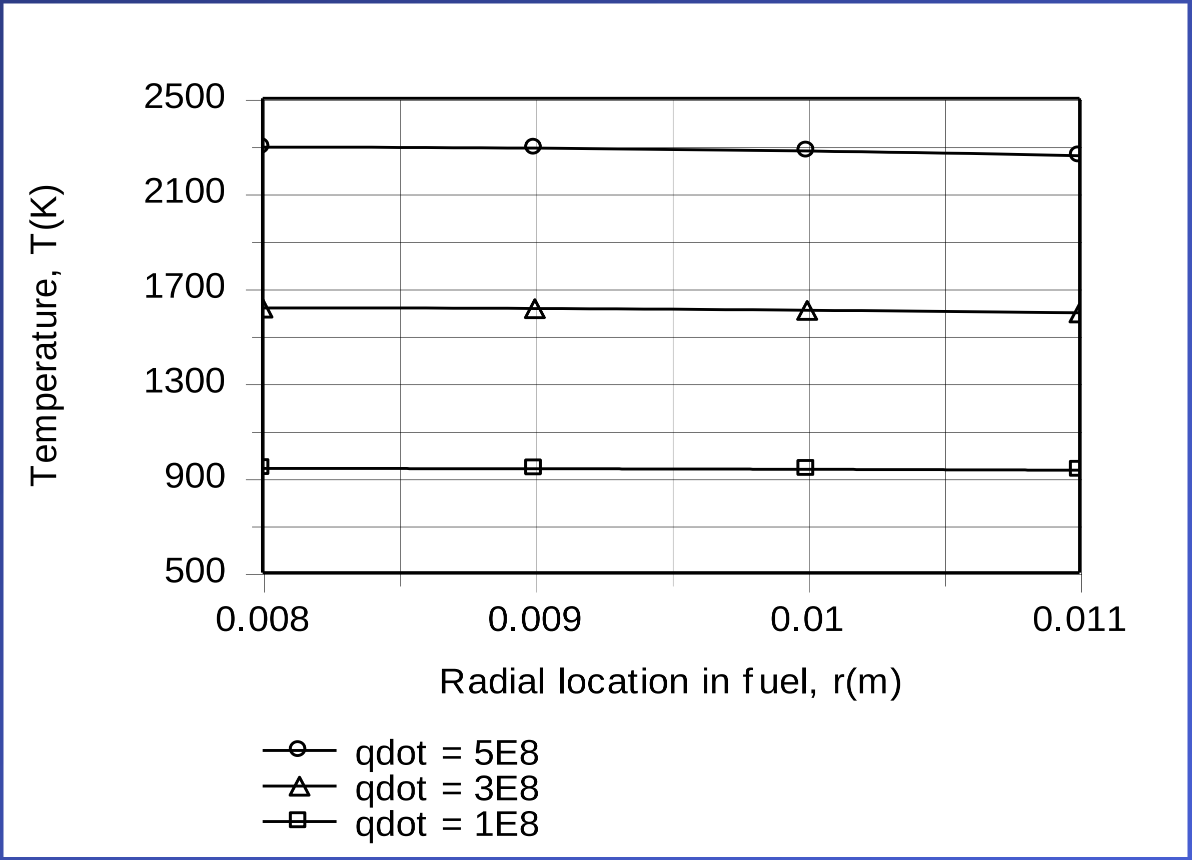
Boundary conditions at *r1* and *r2* are used to determine *T1* and *T2*.

$$r = r_1: \qquad q_1'' = 0 = \frac{\dot{q} r_1}{2} - \frac{k\left[\frac{\dot{q}\, r_2^2}{4k_t}\left(1 - \frac{r_1^2}{r_2^2}\right) + \left(T_2 - T_1\right)\right]}{r_1 \ln\left(r_2 / r_1\right)} \qquad \text{(C.14)}$$

$$r = r_2: \qquad U_2\left(T_2 - T_\infty\right) = \frac{\dot{q}\, r_2}{2} - \frac{k\left[\frac{\dot{q} r_2^2}{4k_t}\left(1 - \frac{r_1^2}{r_2^2}\right) + \left(T_2 - T_1\right)\right]}{r_2 \ln\left(r_2 / r_1\right)} \qquad \text{(C.17)}$$

$$U_2 = \left(A_2' R_{tot}'\right)^{-1} = \left(2\pi r_2 R_{tot}'\right)^{-1} \qquad \text{(3.37)}$$

The following results are obtained for temperature distributions in the graphite.




$\longleftarrow T_{mp} = 2000\text{K}$

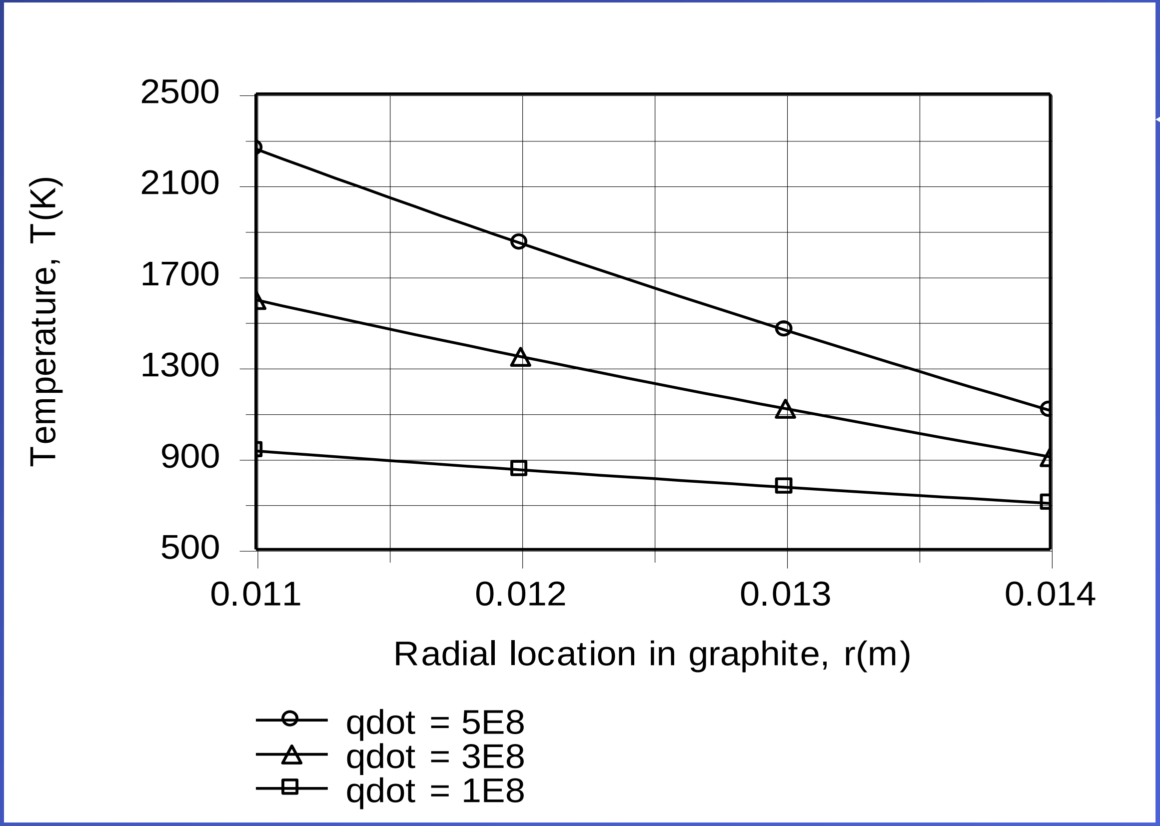
Operation at $\dot{q} = 5\times10^8\,\text{W/m}^3$ is clearly unacceptable since the melting point of thorium would be exceeded. To prevent softening of the material, which would occur below the melting point, the reactor should not be operated much above $\dot{q} = 3\times10^8\,\text{W/m}^3$. The small radial temperature gradients are attributable to the large value of $k_t$.

Using the value of *T2* from the foregoing solution and computing *T3* from the surface condition,

$$q' = \frac{2\pi k_g \left(T_2 - T_3\right)}{\ln\left(r_3 / r_2\right)}$$

the temperature distribution in the graphite is

$$T_g\left(r\right) = \frac{T_2 - T_3}{\ln\left(r_2 / r_3\right)} \ln\left(\frac{r}{r_3}\right) + T_3 \qquad (3.31)$$




$\leftarrow T_{mp} = 2300\text{K}$

Operation at $\dot{q} = 5\times10^{8}\,\text{W/m}^{3}$ is problematic for the graphite. Larger temperature gradients are due to the small value of $k_g$.

**Comments:** (i) What effect would a contact resistance at the thorium/graphite interface have on temperatures in the fuel element and on the maximum allowable value of $\dot{q}$ ? (ii) Referring to the schematic, where might radiation effects be significant? What would be the influence of such effect on temperatures in the fuel element and the maximum allowable value of $\dot{q}$ ?

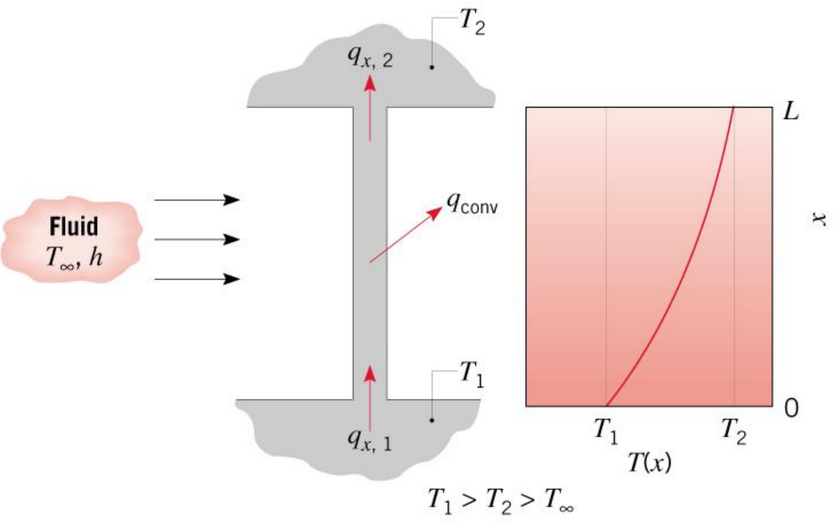
# Extended Surfaces

Chapter Three

Section 3.6

# Nature and Rationale of Extended Surfaces

- An extended surface (also know as a combined conduction-convection system or a fin) is a solid within which heat transfer by conduction is *assumed* to be one dimensional, while heat is also transferred by convection (and/or radiation) from the surface in a direction transverse to that of conduction.




  - If heat is transferred from the surface to the fluid by convection, what surface condition is dictated by the conservation of energy requirement?

  - Why is heat transfer by conduction in the *x*-direction not, in fact, one-dimensional?

– What is the actual functional dependence of the temperature distribution in the solid?

– If the temperature distribution is assumed to be one-dimensional, that is, $T=T(x)$, how should the value of $T$ be interpreted for any $x$ location?

– How does $q_{\text{cond},x}$ vary with $x$?

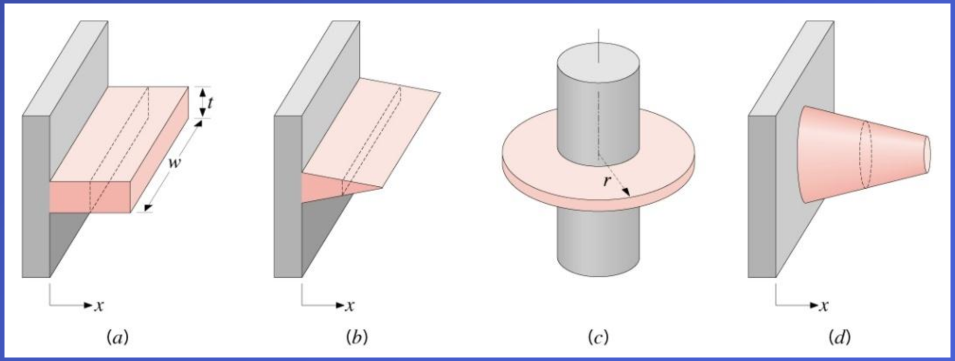
– When may the assumption of one-dimensional conduction be viewed as an excellent approximation?

The thin-fin approximation.

- Extended surfaces may exist in many situations but are commonly used as fins to enhance heat transfer by increasing the surface area available for convection (and/or radiation).

They are particularly beneficial when $h$ is small, as for a gas and natural convection.

- Some typical fin configurations:




Straight fins of (a) uniform and (b) non-uniform cross sections; (c) annular fin, and (d) pin fin of non-uniform cross section.

# The Fin Equation

How is the fin equation derived?

- Assuming one-dimensional, steady-state conduction in an extended surface of constant conductivity $(k)$ and uniform cross-sectional area $(A_c)$, with negligible generation $(\dot{q} = 0)$ and radiation $(q''_{\text{rad}} = 0)$, the *fin equation* is of the form:

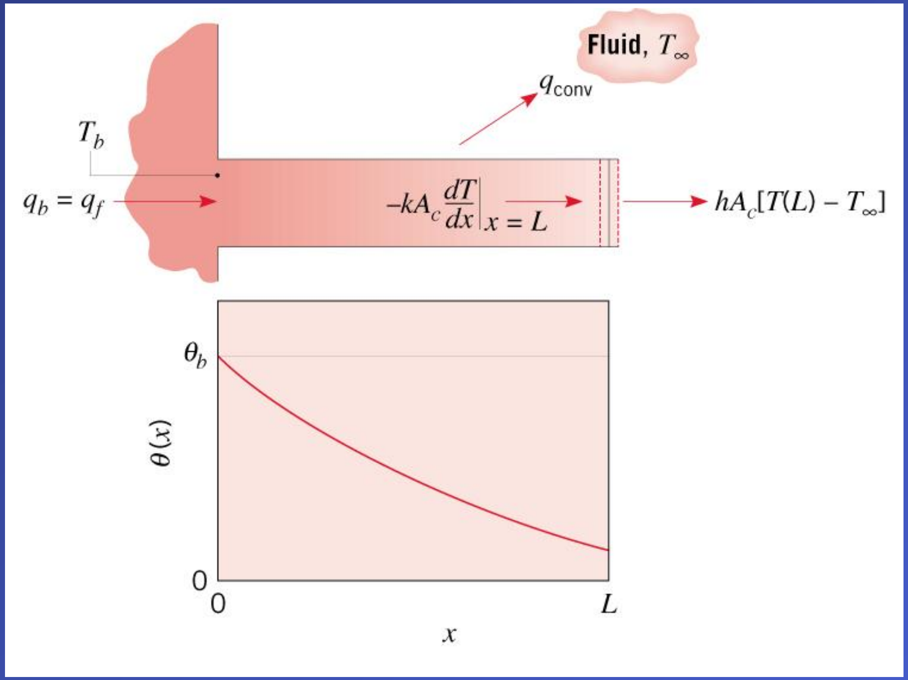
$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0 \qquad (3.67)$$

or, with $m^2 \equiv (hP/kA_c)$ and the reduced temperature $\theta \equiv T - T_\infty$,

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad (3.69)$$

Fin Equation (cont.)

- Solutions (Table 3.4):




Base ($x = 0$) condition

$$\theta(0) = T_b - T_\infty \equiv \theta_b$$

Tip ($x = L$) conditions

A. Convection: $-k\,d\theta/dx|_{x=L} = h\theta(L)$

B. Adiabatic: $d\theta/dx|_{x=L} = 0$

C. Fixed temperature: $\theta(L) = \theta_L$

D. Infinite fin ($mL > 2.65$): $\theta(L) = 0$

- Fin Heat Rate:

$$q_f = -kA_c \frac{d\theta}{dx}\Big|_{x=0} = \int_{A_f} h\theta(x)\,dA_s$$

# Fin Performance Parameters

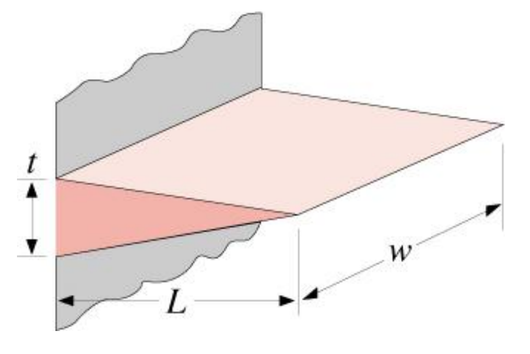
- Fin Efficiency:

$$\eta_f \equiv \frac{q_f}{q_{f,\ \max}} = \frac{q_f}{hA_f\theta_b} \qquad \text{where } 0 \le \eta_f \le 1 \qquad (3.91)$$

How is the efficiency affected by the thermal conductivity of the fin?

Expressions for $\eta_f$ are provided in Table 3.5 for common geometries.

Consider a triangular fin:




$$A_f = 2w\left[L^2 + (t/2)^2\right]^{1/2}$$

$$A_p = (t/2)L$$

$$\eta_f = \frac{1}{mL}\frac{I_1(2mL)}{I_0(2mL)}$$

- Fin Effectiveness:

$$\varepsilon_f \equiv \frac{q_f}{hA_{c,b}\theta_b} \qquad (3.86)$$

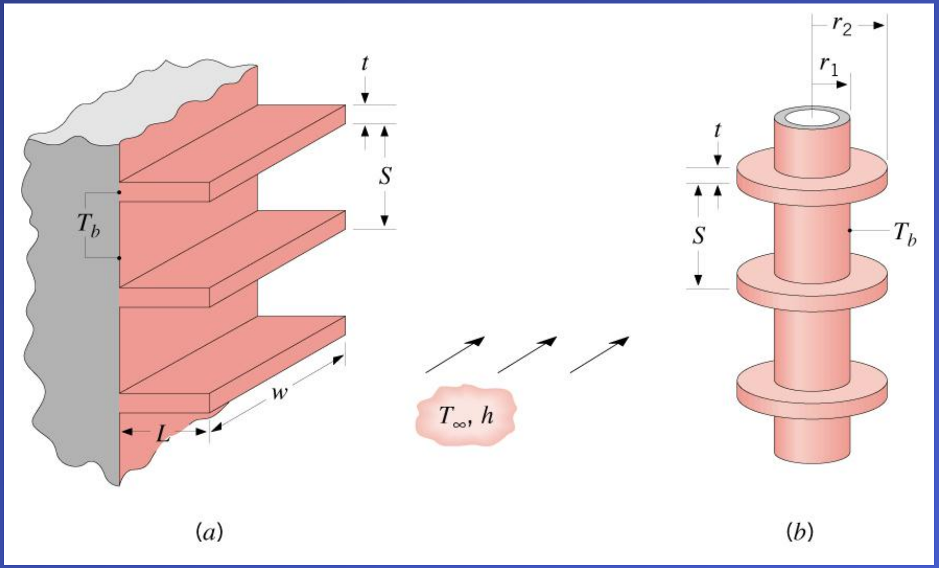
$$\varepsilon_f \uparrow \text{ with } \downarrow h, \uparrow k \text{ and } \downarrow A_c/P$$

- Fin Resistance:

$$R_{t,f} \equiv \frac{\theta_b}{q_f} = \frac{1}{hA_f\eta_f} \qquad (3.97)$$

# Fin Arrays

- Representative arrays of
  (a) rectangular and
  (b) annular fins.




  – Total surface area:

$$A_t = NA_f + A_b \qquad (3.104)$$

Number of fins — Area of exposed base (*prime* surface)

  – Total heat rate:

$$q_t = N\eta_f hA_f\theta_b + hA_b\theta_b \equiv \eta_o hA_t\theta_b = \frac{\theta_b}{R_{t,o}} \qquad (3.105)$$

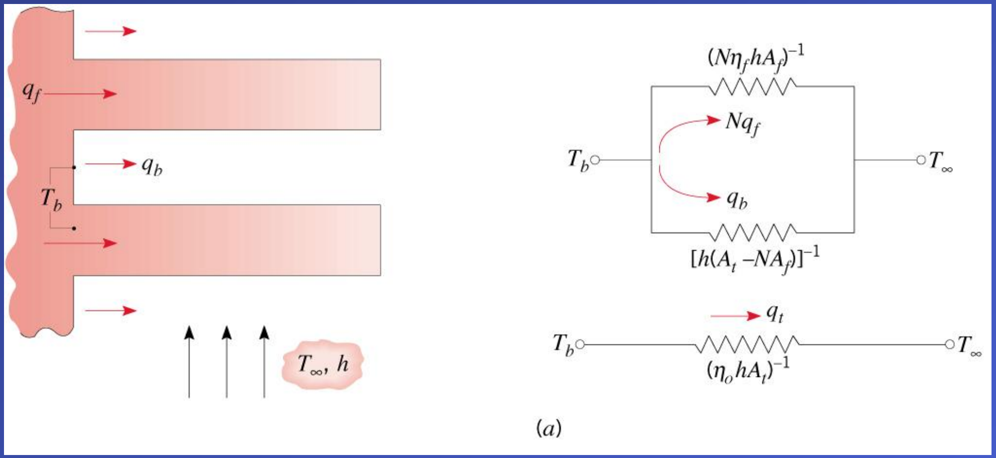
  – Overall surface efficiency and resistance:

$$\eta_o = 1 - \frac{NA_f}{A_t}\left(1-\eta_f\right) \qquad (3.107)$$

$$R_{t,o} = \frac{\theta_b}{q_t} = \frac{1}{\eta_o hA_t} \qquad (3.108)$$

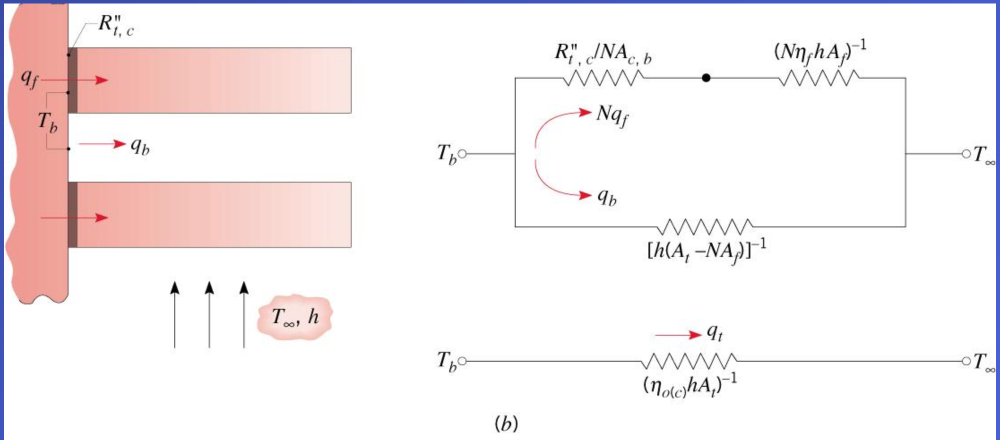
Arrays (Cont.)

- Equivalent Thermal Circuit:



(a)

- Effect of Surface Contact Resistance:



(b)

$$q_t = \eta_{o(c)} hA_t \theta_b = \frac{\theta_b}{R_{t,o(c)}}$$

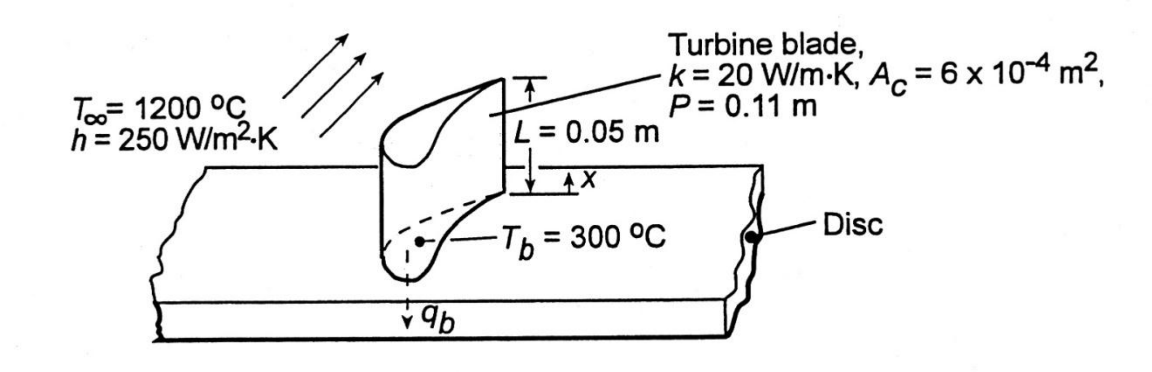
$$\eta_{o(c)} = 1 - \frac{NA_f}{A_t}\left(1 - \frac{\eta_f}{C_1}\right) \quad (3.110a)$$

$$C_1 = 1 + \eta_f hA_f \left(R''_{t,c} / A_{c,b}\right) \quad (3.110b)$$

$$R_{t,o(c)} = \frac{1}{\eta_{o(c)} hA_t} \quad (3.109)$$

Problem 3.126: Assessment of cooling scheme for gas turbine blade.
Determination of whether blade temperatures are less than the maximum allowable value (1050°C) for prescribed operating conditions and evaluation of blade cooling rate.

Schematic:




Assumptions: (1) One-dimensional, steady-state conduction in blade, (2) Constant $k$, (3) Adiabatic blade tip, (4) Negligible radiation.

Analysis: Conditions in the blade are determined by Case B of Table 3.4.

(a) With the maximum temperature existing at $x = L$, Eq. 3.80 yields

$$\frac{T(L) - T_\infty}{T_b - T_\infty} = \frac{1}{\cosh mL}$$

$$m = (hP/kA_c)^{1/2} = \left(250\ \text{W/m}^2\cdot\text{K} \times 0.11\text{m}/20\text{W/m}\cdot\text{K} \times 6\times10^{-4}\text{m}^2\right)^{1/2} = 47.87\ \text{m}^{-1}$$

$$mL = 47.87\ \text{m}^{-1} \times 0.05\ \text{m} = 2.39$$

From Table B.1 (or by calculation), $\cosh mL = 5.51$. Hence,

$$T(L) = 1200^{o}\mathrm{C} + (300 - 1200)^{o}\mathrm{C}/5.51 = 1037^{o}\mathrm{C}$$

and, *subject to the assumption of an adiabatic tip*, the operating conditions are acceptable.

(b) With $M = \left(hPkA_c\right)^{1/2}\theta_b = \left(250\,\mathrm{W/m^2\cdot K} \times 0.11\,\mathrm{m} \times 20\,\mathrm{W/m\cdot K} \times 6\times10^{-4}\,\mathrm{m^2}\right)^{1/2}\left(-900^{o}\mathrm{C}\right) = -517\,\mathrm{W}$,

Eq. 3.81 and Table B.1 yield

$$q_f = M\tanh mL = -517\,\mathrm{W}(0.983) = -508\,\mathrm{W}$$

Hence,

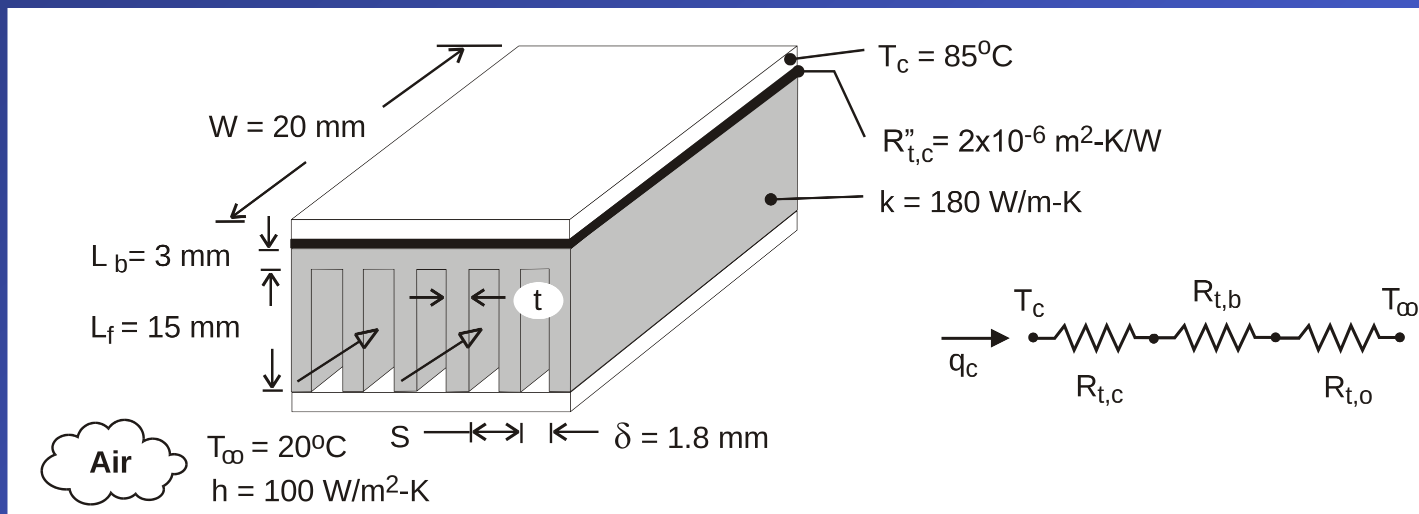
$$q_b = -q_f = 508\,\mathrm{W}$$

<

Comments: Radiation losses from the blade surface contribute to reducing the blade temperatures, but what is the effect of assuming an adiabatic tip condition? Calculate the tip temperature allowing for convection from the gas.

Problem 3.144: Determination of maximum allowable power $q_c$ for a 20 mm × 20 mm electronic chip whose temperature is not to exceed $T_c = 85^\circ C$, when the chip is attached to an air-cooled heat sink with $N = 11$ fins of prescribed dimensions.

Schematic:




Assumptions: (1) Steady-state, (2) One-dimensional heat transfer, (3) Isothermal chip, (4) Negligible heat transfer from top surface of chip, (5) Negligible temperature rise for air flow, (6) Uniform convection coefficient associated with air flow through channels and over outer surface of heat sink, (7) Negligible radiation, (8) Adiabatic fin tips.

Analysis: (a) From the thermal circuit,

$$q_c = \frac{T_c - T_\infty}{R_{tot}} = \frac{T_c - T_\infty}{R_{t,c} + R_{t,b} + R_{t,o}}$$

$$R_{t,c} = R''_{t,c} / W^2 = 2 \times 10^{-6}\ \mathrm{m^2 \cdot K/W} / (0.02\mathrm{m})^2 = 0.005\ \mathrm{K/W}$$

$$R_{t,b} = L_b / k\left(W^2\right) = 0.003\mathrm{m} / 180\ \mathrm{W/m \cdot K}\,(0.02\mathrm{m})^2 = 0.042\ \mathrm{K/W}$$

From Eqs. (3.108), (3.107), and (3.104)

$$R_{t,o} = \frac{1}{\eta_o h A_t},\ \eta_o = 1 - \frac{NA_f}{A_t}\left(1 - \eta_f\right), \qquad A_t = NA_f + A_b$$

$A_f = 2WL_f = 2 \times 0.02\mathrm{m} \times 0.015\mathrm{m} = 6 \times 10^{-4}\ \mathrm{m}^2$

$A_b = W^2 - N(tW) = (0.02\mathrm{m})^2 - 11(0.182 \times 10^{-3}\ \mathrm{m} \times 0.02\mathrm{m}) = 3.6 \times 10^{-4}\ \mathrm{m}^2$

$A_t = 6.96 \times 10^{-3}\ \mathrm{m}^2$

With $mL_f = (2h/kt)^{1/2} L_f = (200\ \mathrm{W/m^2 \cdot K} / 180\ \mathrm{W/m \cdot K} \times 0.182 \times 10^{-3}\mathrm{m})^{1/2}\ (0.015\mathrm{m}) =$ 1.17, $\tanh mL_f = 0.824$ and Eq. (3.94) yields

$$\eta_f = \frac{\tanh mL_f}{mL_f} = \frac{0.824}{1.17} = 0.704$$

$\eta_o = 0.719$,

$R_{t,o} = 2.00$ K/W, and

$$q_c = \frac{(85 - 20)°\mathrm{C}}{(0.005 + 0.042 + 2.00)\ \mathrm{K/W}} = 31.8\ \mathrm{W}$$

<

Comments: The heat sink significantly increases the allowable heat dissipation. If it were not used and heat was simply transferred by convection from the surface of the chip with $h = 100\ \text{W/m}^2 \cdot \text{K}, R_{\text{tot}} = 2.05\ \text{K/W}$ from Part (a) would be replaced by $R_{\text{conv}} = 1/hW^2 = 25\ \text{K/W}$, yielding $q_c = 2.60\ \text{W}$.

From Equations 3.125 and 3.126,

$$q_1 = \frac{1}{R_{t,\text{cond,mod}}}(T_1 - T_2) + IS_{p\text{-}n,\text{eff}}T_1 - I^2 R_{e,\text{eff}} = \frac{(T_1 - T_2)}{1.736\ \text{K/W}} + I \times 0.1435\ \text{V/K} \times T_1 - I^2 \times 4\ \Omega \qquad (2)$$

$$q_2 = \frac{1}{R_{t,\text{cond,mod}}}(T_1 - T_2) + IS_{p\text{-}n,\text{eff}}T_2 + I^2 R_{e,\text{eff}} = \frac{(T_1 - T_2)}{1.736\ \text{K/W}} + I \times 0.1435\ \text{V/K} \times T_2 + I^2 \times 4\ \Omega \qquad (3)$$

For heat transfer by radiation to deep space,

$$q_2 = h_r W^2 (T_2 - T_{\text{sur}}) = h_r \times (0.054\ \text{m})^2 \times (T_2 - 4\ \text{K}) \qquad (4)$$

where,

$$h_r = \varepsilon\sigma(T_2 + T_{\text{sur}})(T_2^2 + T_{\text{sur}}^2) = 0.93 \times 5.67 \times 10^{-8}\ \text{W/m}^2 \cdot \text{K} \times (T_2 + 4\ \text{K}) \times (T_2^2 + (4\ \text{K})^2) \qquad (5)$$

The electric power produced by all $M = 80$ modules, $P_{\text{tot}}$, is related to the power generated in each module, $P_N$, and the load resistance, $R_{\text{e,load}}$

$$P_{\text{tot}} = MP_N = M\left[IS_{p\text{-}n,\text{eff}}(T_1 - T_2) - 2I^2 R_{e,\text{eff}}\right] = I^2 R_{e,\text{load}}$$

or,

$$80\left[I \times 0.1435\ \text{V/K} \times (T_1 - T_2) - 2I^2 \times 4\ \Omega\right] = I^2 \times 250\ \Omega \qquad (6)$$

$\dot{E}_g$

# Problem: Thermoelectric power generation with radiation cooling (cont.)

Upon specification of $\dot{E}_g$ Equation (1) may be solved for $q_1$. Equations (2) through (6) may then be solved simultaneously for $T_1$, $T_2$, $I$, $q_2$, and $h_r$.

| $\dot{E}_g$ (kW) | $I$(A) | $P_{tot}$ (W) | $T_2$ (K) | $\eta = P_N / q_1$ |
|---|---|---|---|---|
| 1 | 0.10 | 2.63 | 534 | 0.0026 |
| 10 | 0.67 | 114 | 947 | 0.011 |
| 100 | 3.99 | 3990 | 1671 | 0.04 |

<

COMMENTS: (1) The temperature for the highest thermal energy generation rate is unacceptably high. (2) The electric power generated is relatively high, but the conversion efficiency, $\eta$, is low. The efficiency increases with generation rate because of larger temperature differences across the modules, which are $\Delta T$ = 8, 52, and 310 K for the low, medium, and high generation rates, respectively. (3) What steps might be taken to increase $\Delta T$ and, in turn, increase the conversion efficiency?

If the molecule-surface interaction and corresponding resistance is neglected, the conduction heat rate is determined from

$$q_x = \frac{kA}{L}\left(T_{s,1} - T_{s,2}\right) \qquad (1)$$

The actual conduction heat transfer rates and conduction heat transfer rates calculated from Equation 1 are compared below.

| $L$ | $L/\lambda_{mfp}$ | $q_x$ (actual) | $q_x$ (Equation 1) |
|---|---|---|---|
| 1 mm | 15,000 | 0.0263 W | 0.0263 W |
| 1 μm | 15 | 21.09 W | 26.3 W |
| 10 nm | 0.15 | 102.3 W | 2632 W |

COMMENTS: For relatively large plate spacing, molecule-solid resistances may be safely neglected. However, as $L/\lambda_{mfp}$ becomes smaller, such resistances may become important ($L$ = 1 μm) or dominant ($L$ = 10 nm).