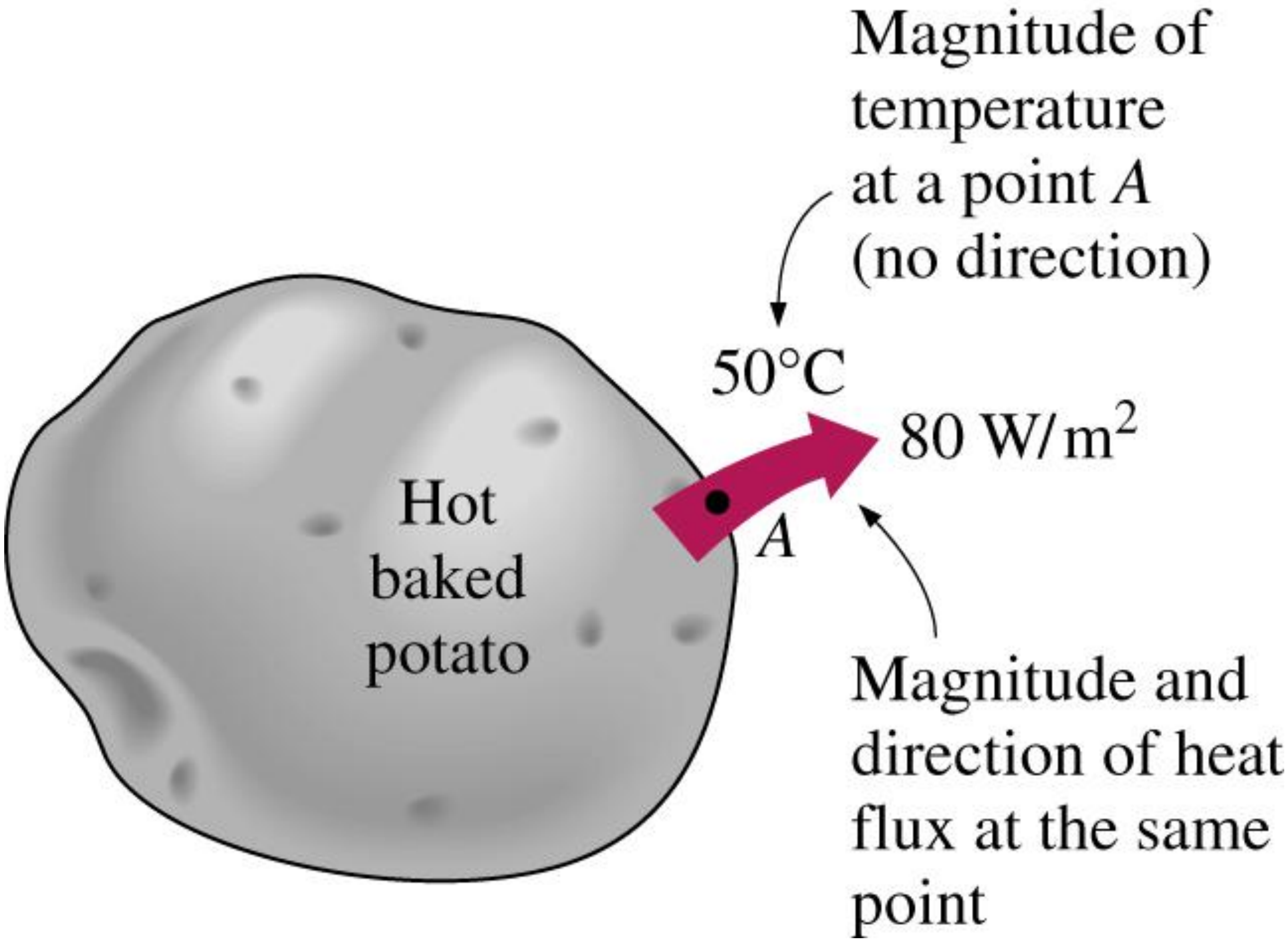
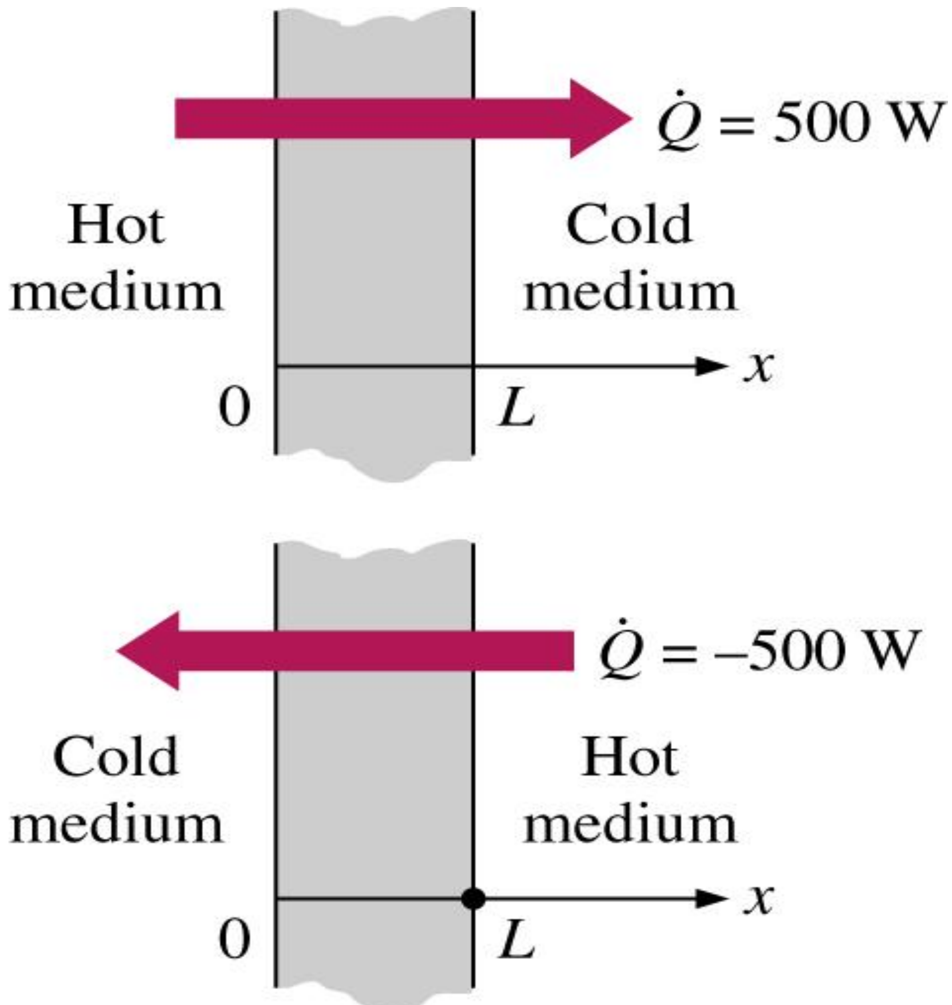
Heat diffusion Equations

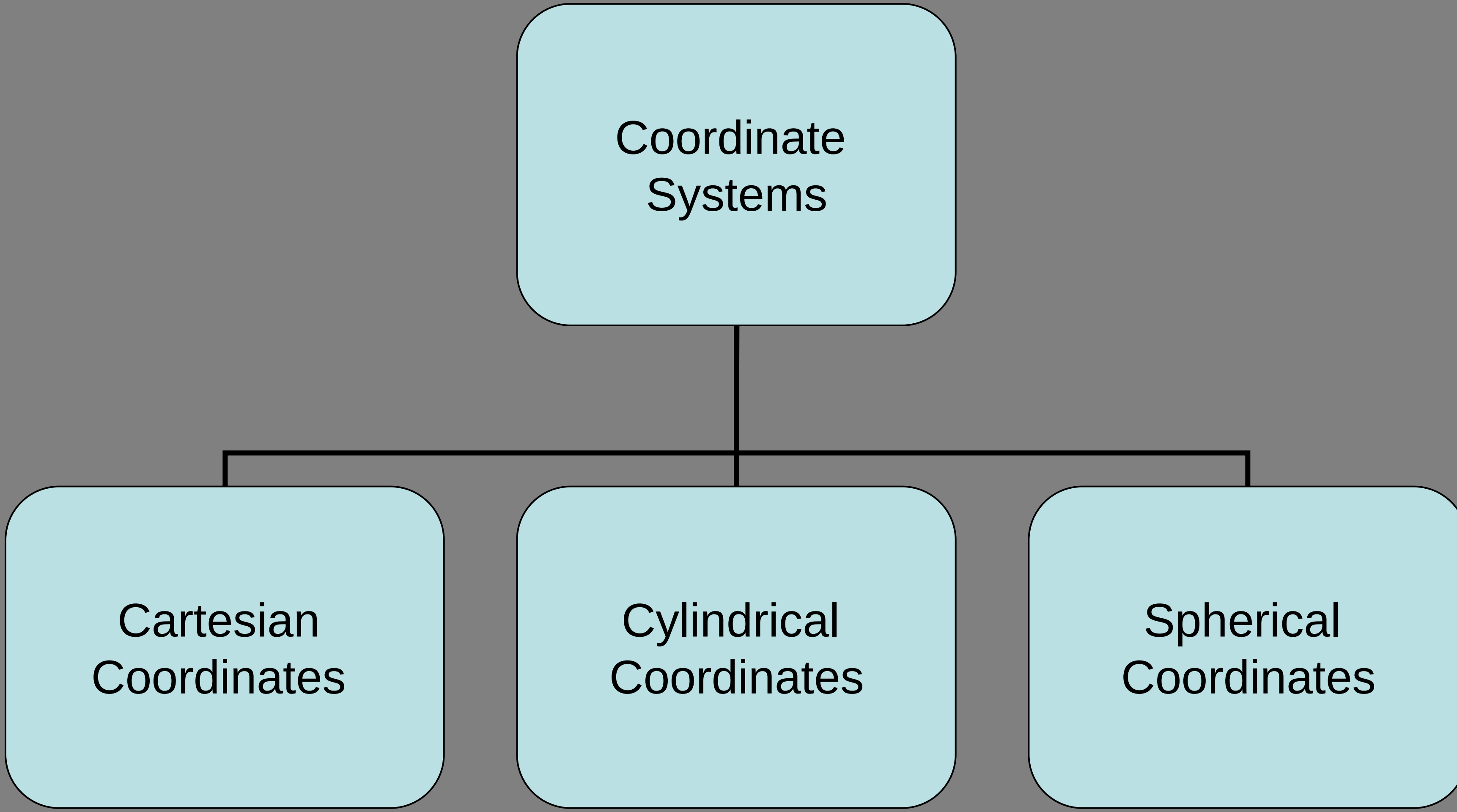
- Equations of heat diffusion can be obtained by applying the energy balance equation over a differential element or a differential control volume.
- Solving these differential equations subjected to certain boundary conditions will lead to the temperature distribution in a medium.
- Once this distribution is known, the conduction heat flux at any point in the medium or on its surface could be computed from Fourier’s law.

Heat transfer has direction plus magnitude and therefore it is a vector quantity

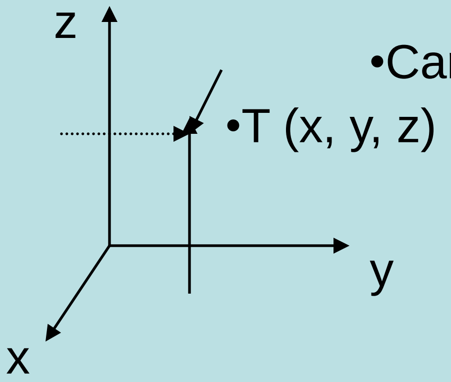


Indicating direction for heat transfer (+ve in the +ve direction and –ve in the –ve direction)

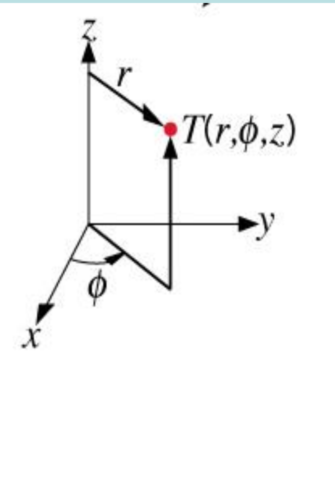



Coordinate Systems
Cartesian Coordinates
Cylindrical Coordinates
Spherical Coordinates

Coordinate systems

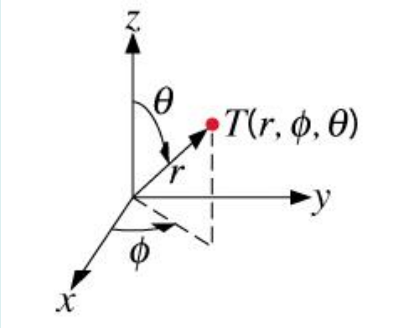


- Cartesian coordinates: T (x, y, z)

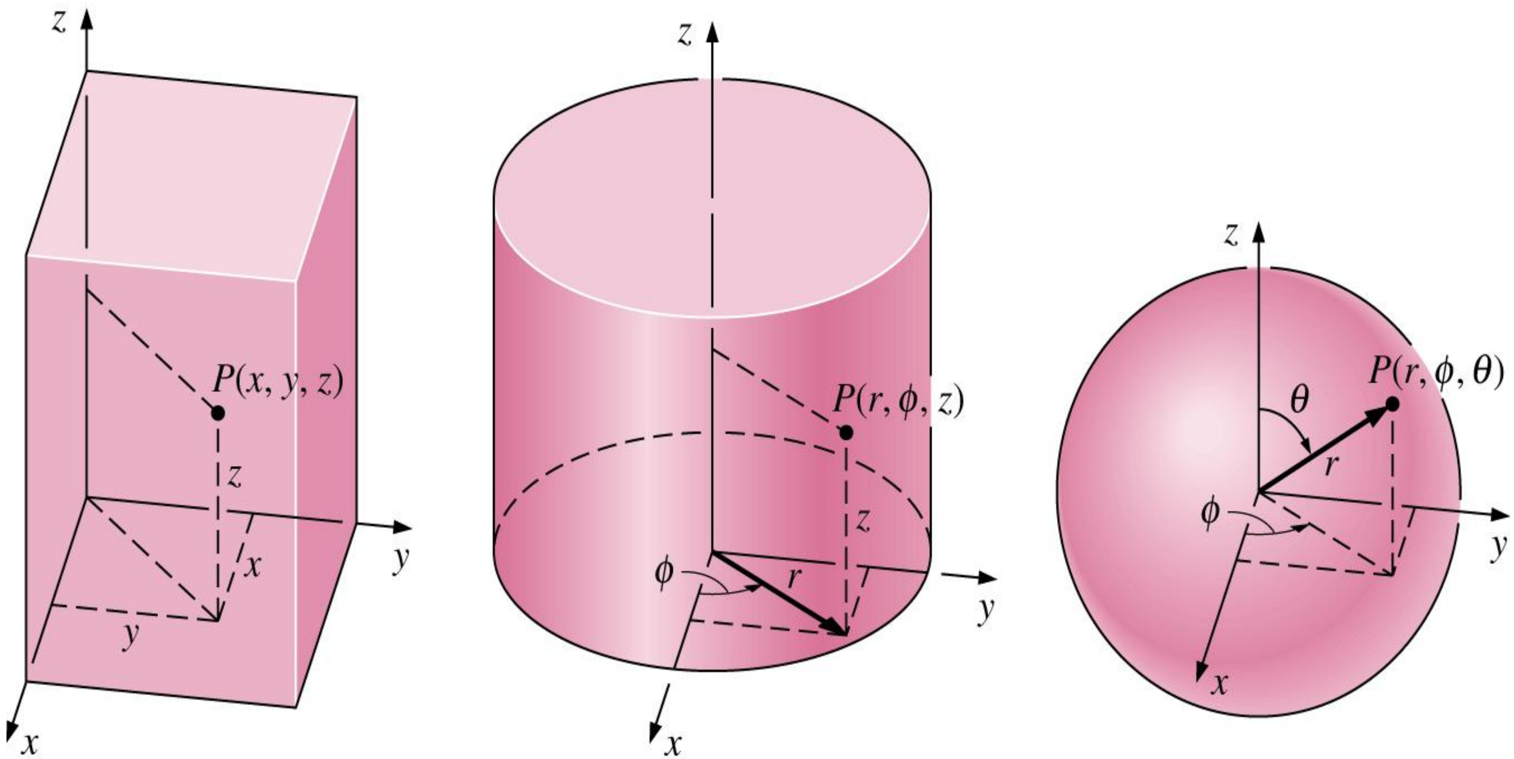


- Cylindrical Coordinates: $T\left(r,\phi,z\right)$

- Spherical Coordinates: $T\left(r,\phi,\theta\right)$

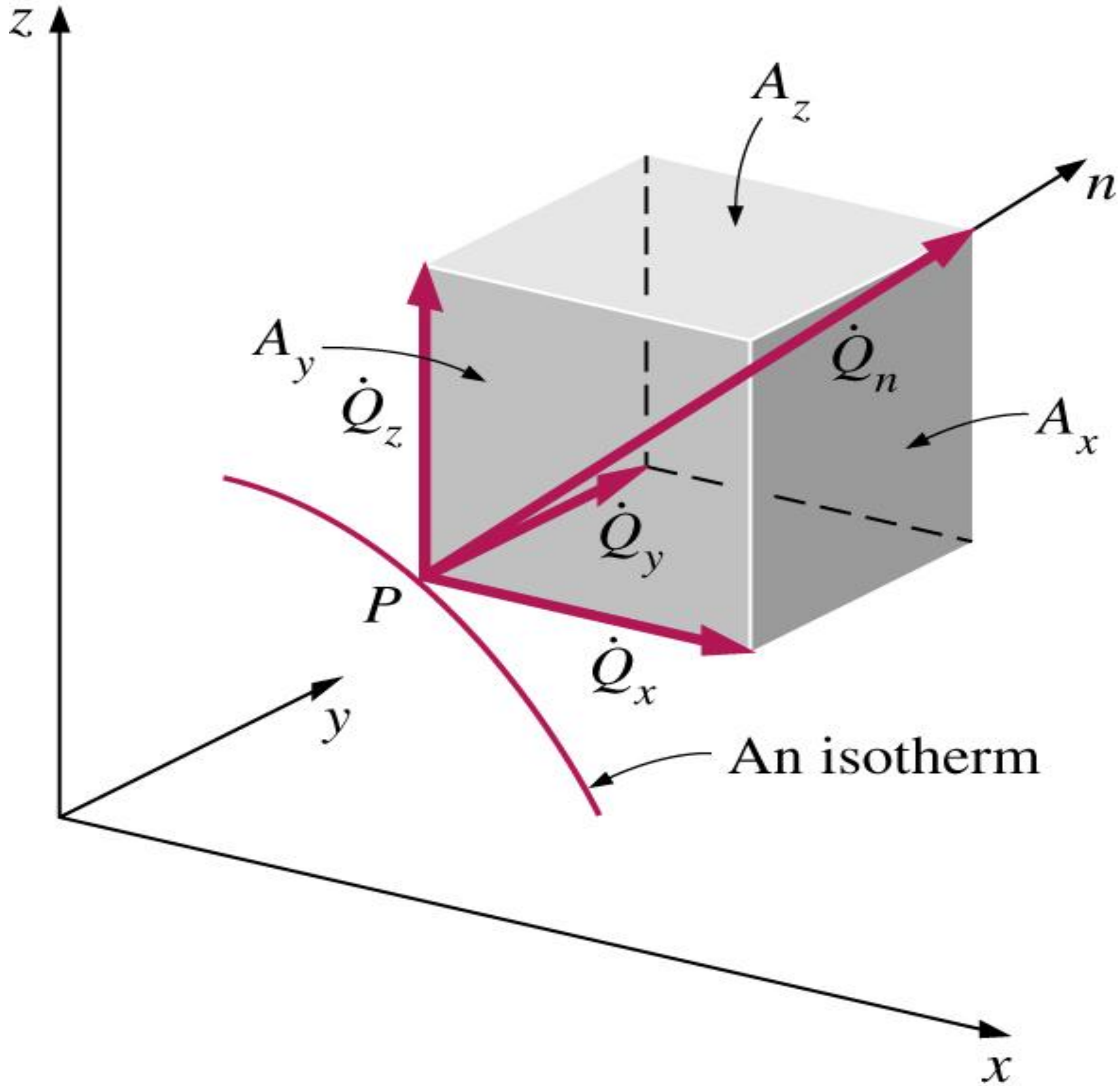


The various distance and angles involved when describing the location of a point in different coordinate system



(*a*) Rectangular coordinates (*b*) Cylindrical coordinates (*c*) Spherical coordinates

The heat transfer vector is always normal to an isothermal surface and can be resolved into its components like any



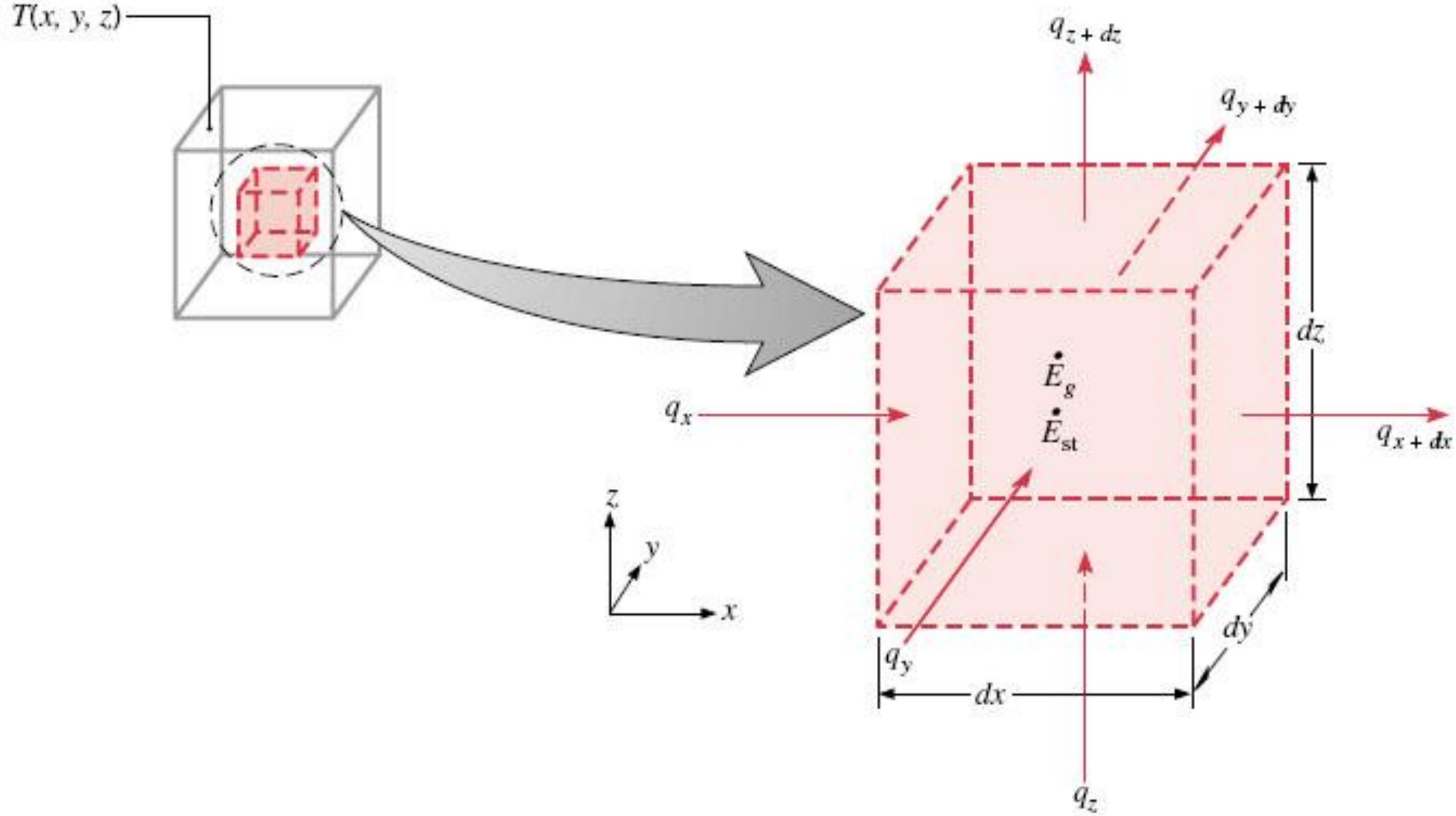


FIGURE 2.11 Differential control volume, $dx\ dy\ dz$, for conduction analysis in Cartesian coordinates.

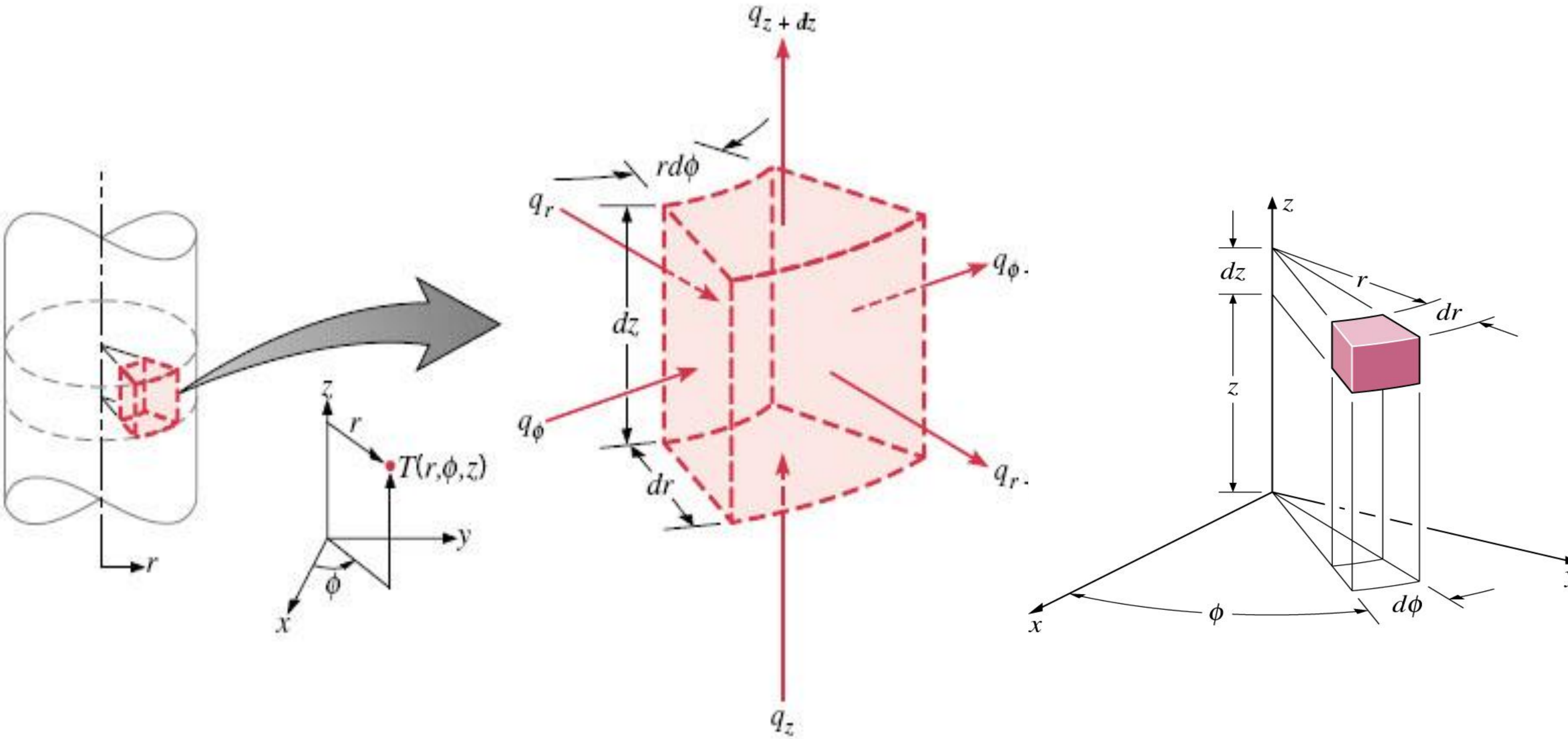


FIGURE 2.12 Differential control volume, $dr \cdot r\,d\phi \cdot dz$, for conduction analysis in cylindrical coordinates (r, ϕ, z).

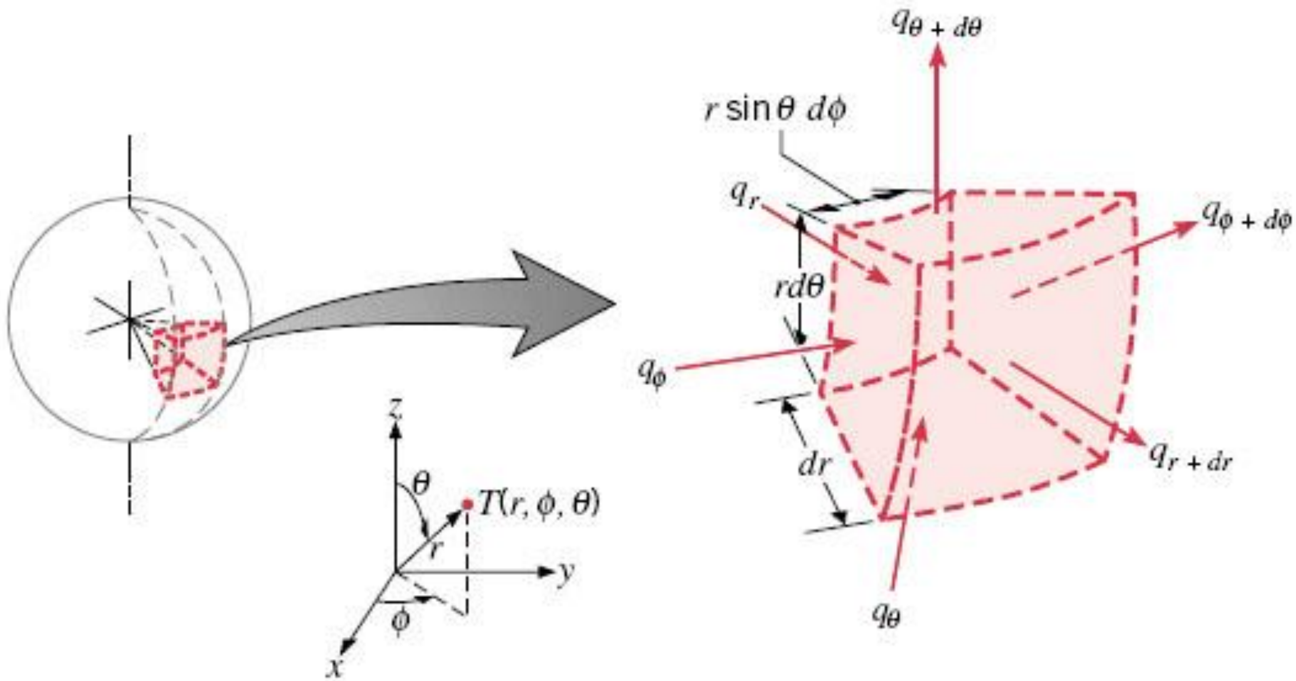


FIGURE 2.13 Differential control volume, $dr \cdot r \sin\theta \, d\phi \cdot r \, d\theta$, for conduction analysis in spherical coordinates (r, ϕ, θ).

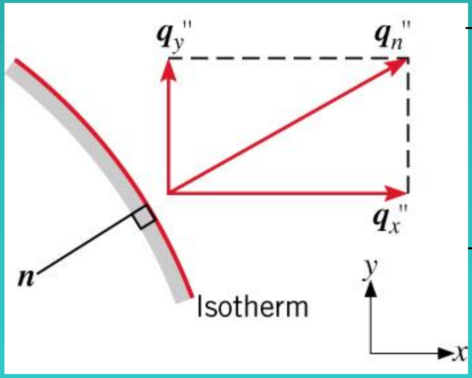
Fourier's Law

- A rate equation that allows determination of the conduction heat flux from knowledge of the temperature distribution in a medium
- Its most general (vector) form for multidimensional conduction is:

$$\vec{q}'' = -k\,\vec{\nabla} T$$

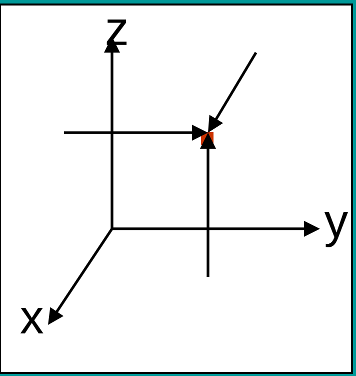
Implications:

- Heat transfer is in the direction of decreasing temperature (basis for minus sign).



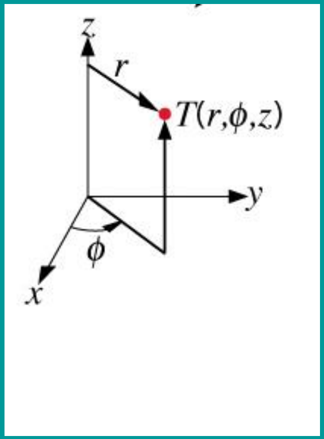
- Fourier's Law serves to define the thermal conductivity of the medium $\left(k \equiv -\vec{q}''/\vec{\nabla} T \right)$
- Direction of heat transfer is perpendicular to lines of constant temperature (isotherms).
- Heat flux vector may be resolved into orthogonal components.

Heat Flux Components



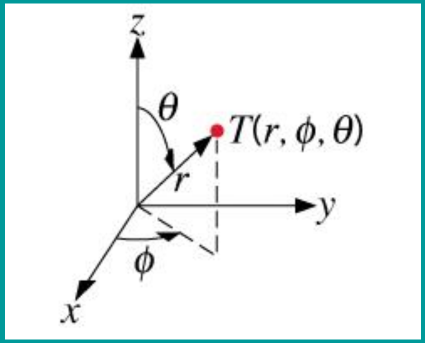
- Cartesian Coordinates: $T(x,y,z)$

$$\vec{q}'' = \underbrace{-k\frac{\partial T}{\partial x}\vec{i}}_{q''_x} \underbrace{-k\frac{\partial T}{\partial y}\vec{j}}_{q''_y} \underbrace{-k\frac{\partial T}{\partial z}\vec{k}}_{q''_z} \qquad (2.3)$$



- Cylindrical Coordinates: $T(r,\phi,z)$

$$\vec{q}'' = \underbrace{-k\frac{\partial T}{\partial r}\vec{i}}_{q''_r} \underbrace{-k\frac{\partial T}{r\partial \phi}\vec{j}}_{q''_\phi} \underbrace{-k\frac{\partial T}{\partial z}\vec{k}}_{q''_z} \qquad (2.22)$$

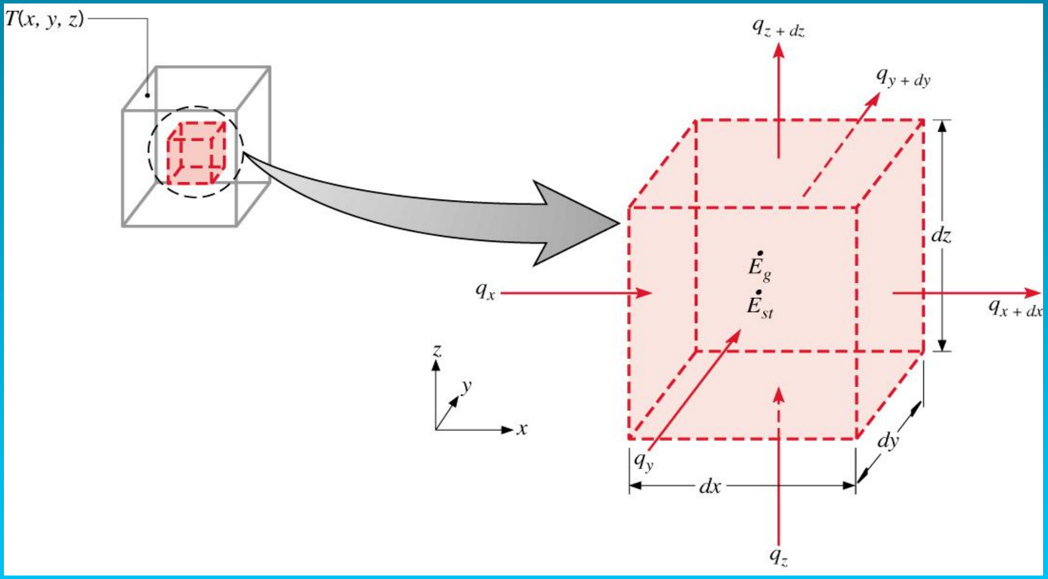


- Spherical Coordinates: $T(r,\phi,\theta)$

$$\vec{q}'' = \underbrace{-k\frac{\partial T}{\partial r}\vec{i}}_{q''_r} \underbrace{-k\frac{\partial T}{r\partial \theta}\vec{j}}_{q''_\theta} \underbrace{-k\frac{\partial T}{r\sin\theta\,\partial \phi}\vec{k}}_{q''_\phi} \qquad (2.25)$$

The Heat Equation

- A differential equation whose solution provides the temperature distribution in a stationary medium.
- Based on applying conservation of energy to a differential control volume through which energy transfer is exclusively by conduction.
- Cartesian Coordinates:

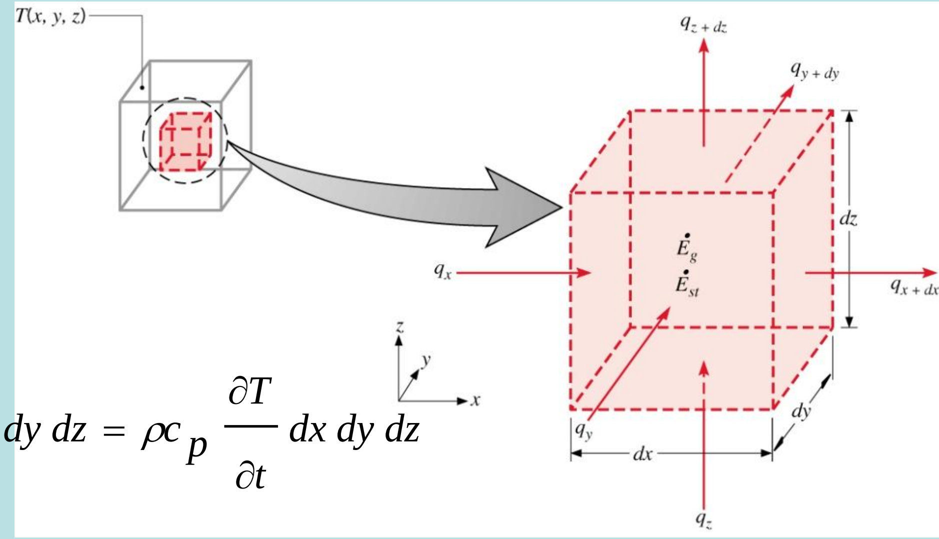


$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t} \qquad (2.17)$$

Net transfer of thermal energy into the control volume (inflow-outflow)

Thermal energy generation

Change in thermal energy storage



Energy Balance:

$E_{in} - E_{out} + E_g = E_{st}$

$$q_x + q_y + q_z - q_{x+dx} - q_{y+dy} - q_{z+dz} + \dot{q}\, dx\, dy\, dz = \rho c_p \frac{\partial T}{\partial t} dx\, dy\, dz$$

$$q_x + q_y + q_z - (q_x + \frac{\partial q_x}{\partial x} dx) - (q_y + \frac{\partial q_y}{\partial y} dy) - (q_z + \frac{\partial q_z}{\partial z} dz) + \dot{q}\, dx\, dy\, dz = \rho c_p \frac{\partial T}{\partial t} dx\, dy\, dz$$

$$-\frac{\partial q_x}{\partial x} dx - \frac{\partial q_y}{\partial y} dy - \frac{\partial q_z}{\partial z} dz + \dot{q}\, dx\, dy\, dz = \rho c_p \frac{\partial t}{\partial t} dx\, dy\, dz$$

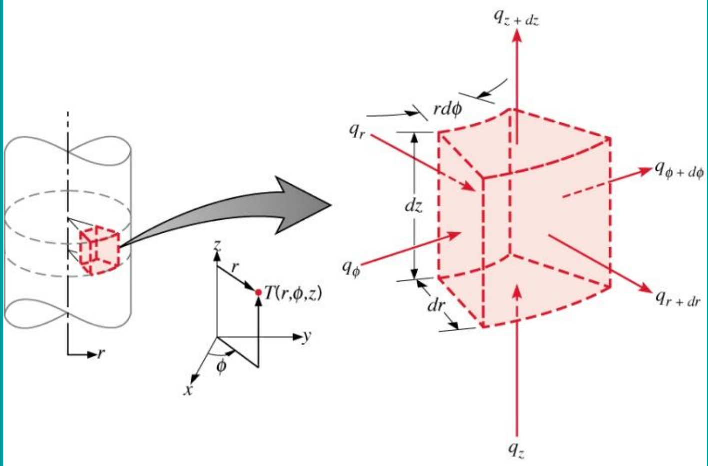
Use Fourier's Law

$$q_x = -k\, dy\, dz \frac{\partial T}{\partial x} \quad , \quad q_y = -k\, dx\, dz \frac{\partial T}{\partial y} \quad , \quad q_z = -k\, dy\, dx \frac{\partial T}{\partial z}$$

$\therefore$ *After substitution and dividing out the volume dx dy dz, we get*

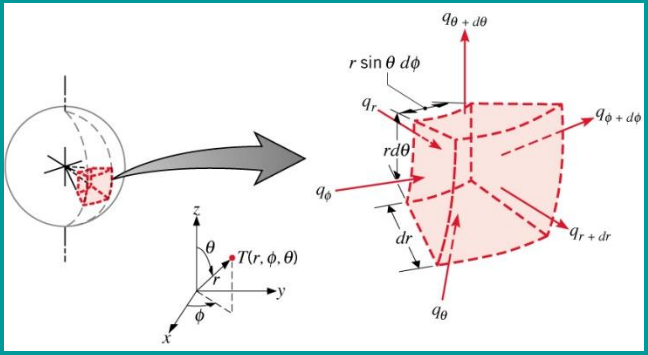
$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

- Cylindrical Coordinates:



$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t} \qquad (2.24)$$

- Spherical Coordinates:



$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t} \qquad (2.27)$$

- One-Dimensional Conduction in a Planar Medium with Constant Properties and No Generation

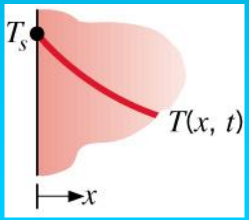
$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\alpha \equiv \frac{k}{\rho c_p} \rightarrow$$ thermal diffusivity of the medium

Boundary and Initial Conditions

- For transient conduction, heat equation is first order in time, requiring specification of an initial temperature distribution: $T(x,t)_{t=0} = T(x,0)$
- Since heat equation is second order in space, two boundary conditions must be specified. Some common cases:

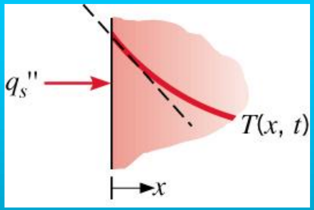
Constant Surface Temperature:



$$T(0,t) = T_s$$

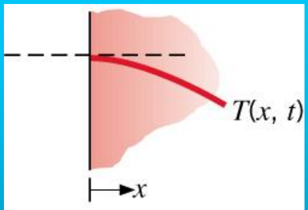
Constant Heat Flux:

Applied Flux



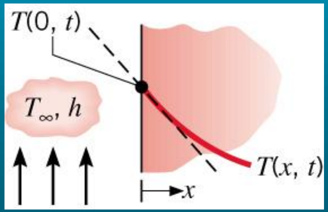
$$-k \frac{\partial T}{\partial x}\Big|_{x=0} = q_s''$$

Insulated Surface



$$\frac{\partial T}{\partial x}\Big|_{x=0} = 0$$

Convection



$$-k \frac{\partial T}{\partial x}\Big|_{x=0} = h\left[T_\infty - T(0,t)\right]$$