Extended Surfaces

Heat Transfer rate; Temp. distribution; Effectiveness; Efficiency

Extended Surfaces

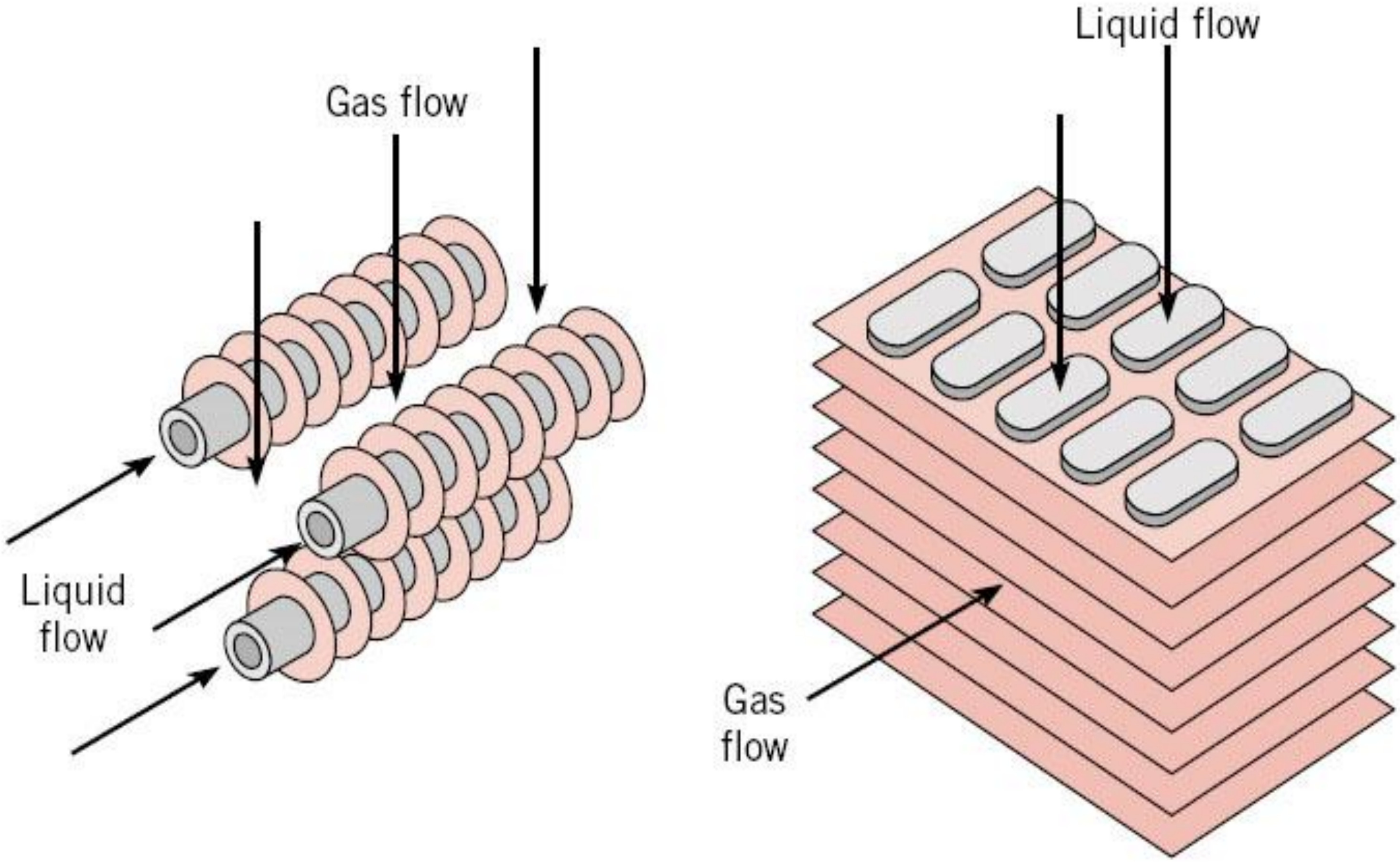


FIGURE 3.13 Schematic of typical finned-tube heat exchangers.

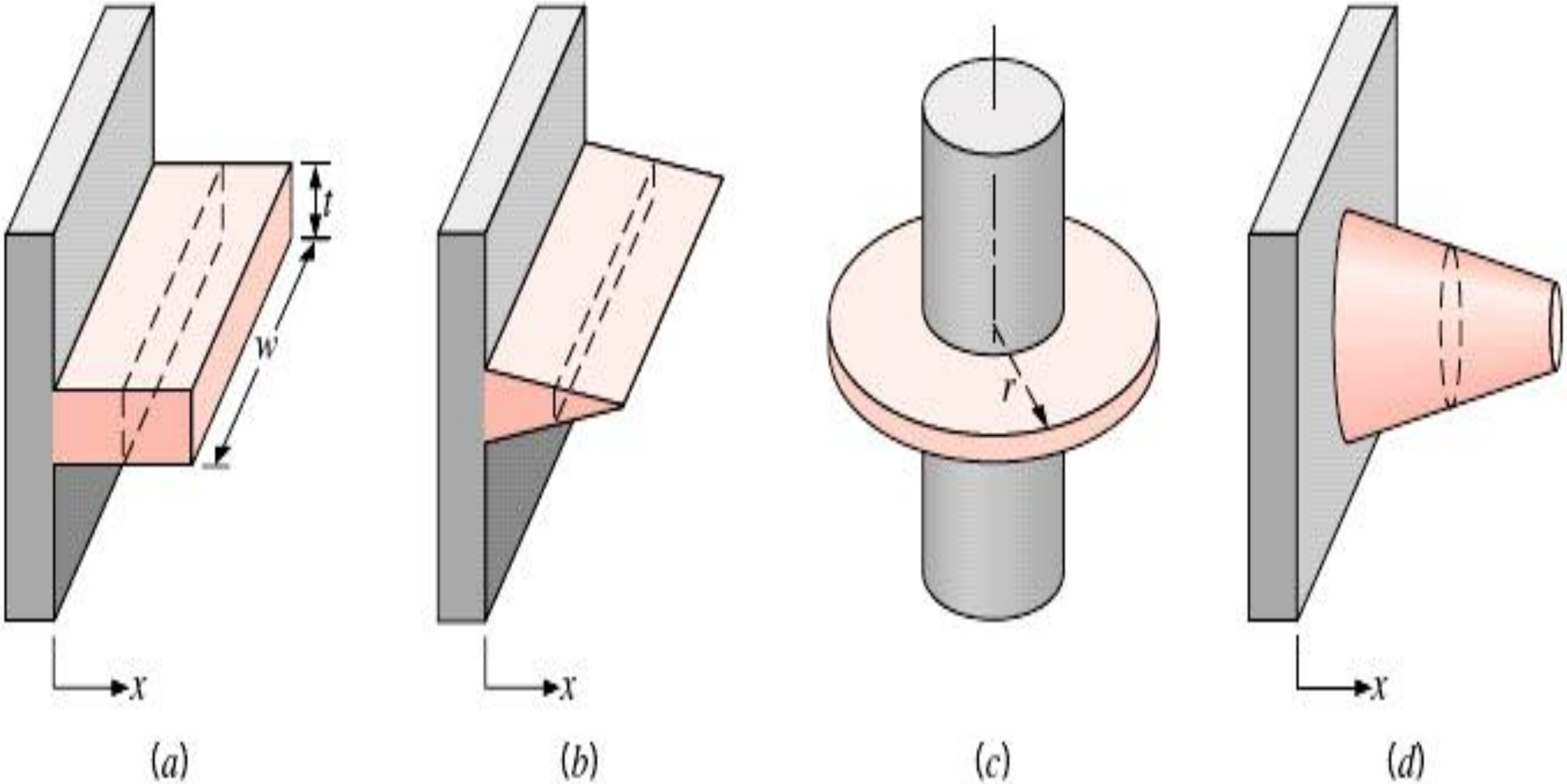


FIGURE 3.14 Fin configurations. (*a*) Straight fin of uniform cross section. (*b*) Straight fin of nonuniform cross section. (*c*) Annular fin. (*d*) Pin fin.

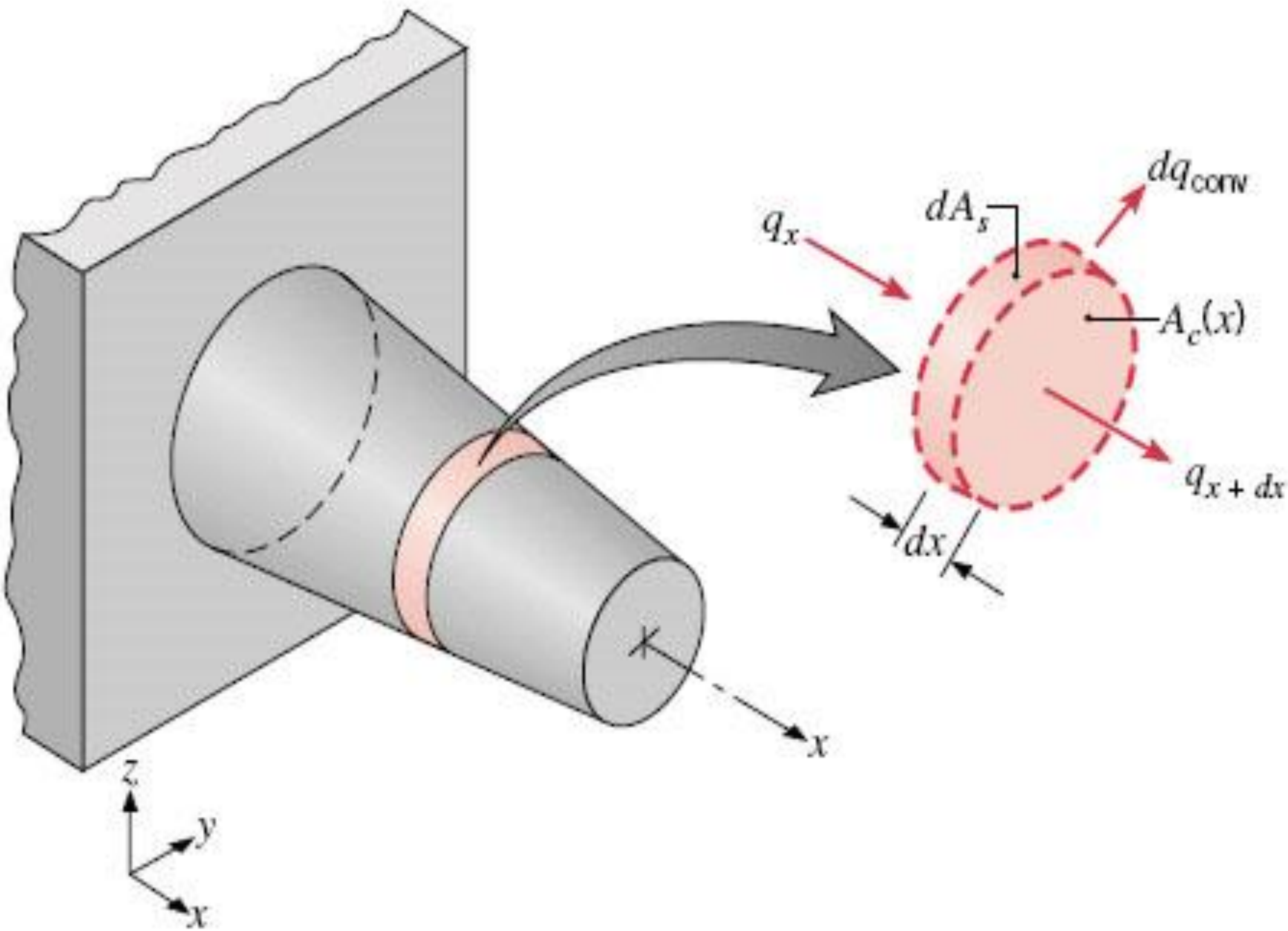


FIGURE 3.15 Energy balance for an extended surface.

General Energy equation for fin

Applying the conservation of energy requirement, Equation 1.11c, to the differential element of Figure 3.15, we obtain

$$q_x = q_{x+dx} + dq_{conv} \tag{3.56}$$

From Fourier's law we know that

$$q_x = -kA_c \frac{dT}{dx} \tag{3.57}$$

where A_c is the *cross-sectional* area, which may vary with x. Since the conduction heat rate at $x + dx$ may be expressed as

$$q_{x+dx} = q_x + \frac{dq_x}{dx} dx \tag{3.58}$$

it follows that

$$q_{x+dx} = -kA_c \frac{dT}{dx} - k \frac{d}{dx}\left(A_c \frac{dT}{dx}\right) dx \tag{3.59}$$

The convection heat transfer rate may be expressed as

$$dq_{\text{conv}} = h\, dA_s (T - T_\infty) \tag{3.60}$$

where dA_s is the *surface* area of the differential element. Substituting the foregoing rate equations into the energy balance, Equation 3.56, we obtain

$$\frac{d}{dx}\left(A_c \frac{dT}{dx}\right) - \frac{h}{k}\frac{dA_s}{dx}(T - T_\infty) = 0$$

or

$$\boxed{\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx} - \left(\frac{1}{A_c}\frac{h}{k}\frac{dA_s}{dx}\right)(T - T_\infty) = 0} \tag{3.61}$$

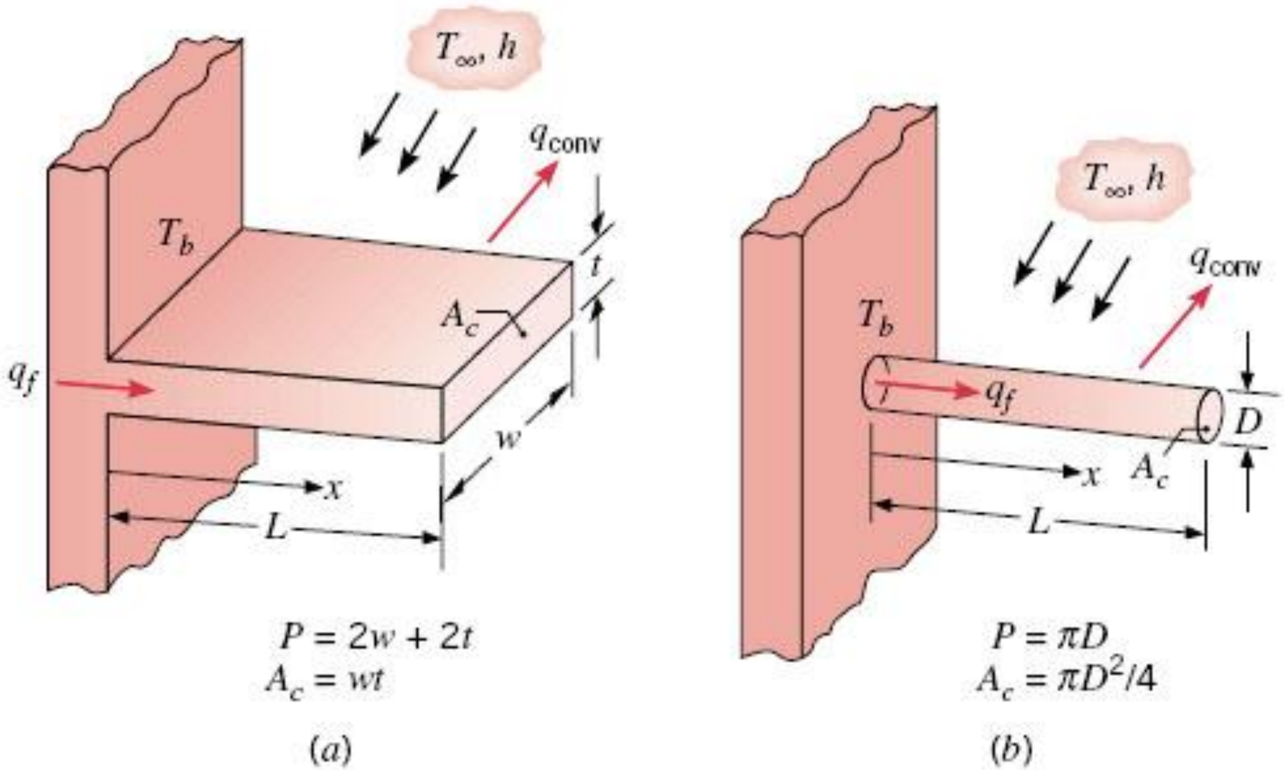


FIGURE 3.16 Straight fins of uniform cross section. (*a*) Rectangular fin. (*b*) Pin fin.

Fins of Uniform Cross-Sectional Area

To solve Equation 3.61 it is necessary to be more specific about the geometry. We begin with the simplest case of straight rectangular and pin fins of uniform cross section (Figure 3.16). Each fin is attached to a base surface of temperature $T(0) = T_b$ and extends into a fluid of temperature T_∞.

For the prescribed fins, A_c is a constant and $A_s = Px$, where A_s is the surface area measured from the base to x and P is the fin perimeter. Accordingly, with $dA_c/dx = 0$ and $dA_s/dx = P$, Equation 3.61 reduces to

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0 \qquad (3.62)$$

To simplify the form of this equation, we transform the dependent variable by defining an *excess temperature* θ as

$$\theta(x) \equiv T(x) - T_\infty \qquad (3.63)$$

where, since T_∞ is a constant, $d\theta/dx = dT/dx$. Substituting Equation 3.63 into Equation 3.62, we then obtain

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad (3.64)$$

where

$$m^2 \equiv \frac{hP}{kA_c} \qquad (3.65)$$

Equation 3.64 is a linear, homogeneous, second-order differential equation with constant coefficients. Its general solution is of the form

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx} \qquad (3.66)$$

To evaluate the constants C_1 and C_2 of Equation 3.66, it is necessary to specify appropriate boundary conditions. One such condition may be specified in terms of the temperature at the *base* of the fin ($x = 0$)

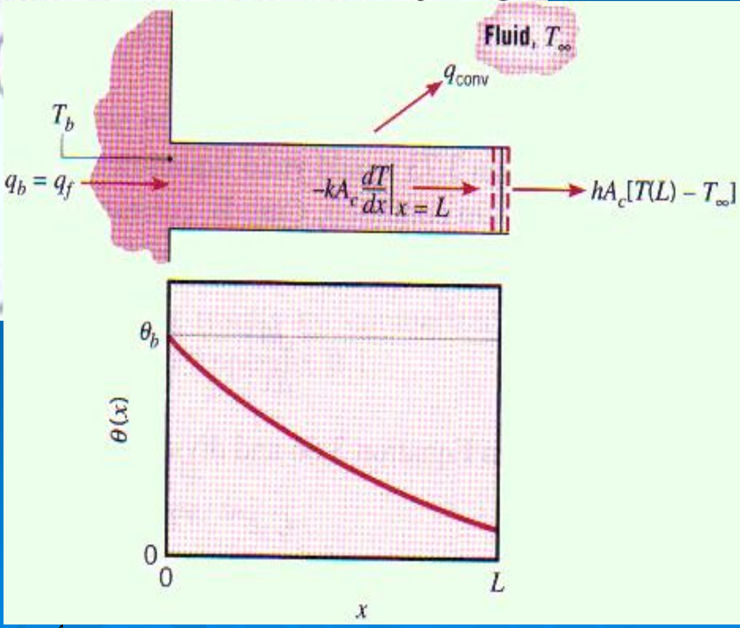
$$\theta(0) = T_b - T_\infty \equiv \theta_b \qquad (3.67)$$

The second condition, specified at the fin tip ($x = L$), may correspond to one of four different physical situations.

The first condition, Case A, considers convection heat transfer from the fin tip. Applying an energy balance to a control surface about this tip (

We obtain

$$hA_c[T(L) - T_\infty] = -kA_c \left.\frac{dT}{dx}\right|_{x=L}$$



or

$$h\theta(L) = -k\left.\frac{d\theta}{dx}\right|_{x=L} \qquad (3.68)$$

Substituting Equation 3.66 into Equations 3.67 and 3.68, we obtain, respectively,

$$\theta_b = C_1 + C_2 \tag{3.69}$$

and

$$h(C_1 e^{mL} + C_2 e^{-mL}) = km(C_2 e^{-mL} - C_1 e^{mL})$$

Solving for C_1 and C_2, it may be shown, after some manipulation, that

Temp. Distribution

See also fig 3.17

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL} \tag{3.70}$$

Heat rate; applying Fourier's Law at the base

$$q_f = q_b = -kA_c \left.\frac{dT}{dx}\right|_{x=0} = -kA_c \left.\frac{d\theta}{dx}\right|_{x=0} \tag{3.71}$$

Hence, knowing the temperature distribution, $\theta(x)$, q_f may be evaluated, giving

$$q_f = \sqrt{hPkA_c}\,\theta_b \frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL} \tag{3.72}$$

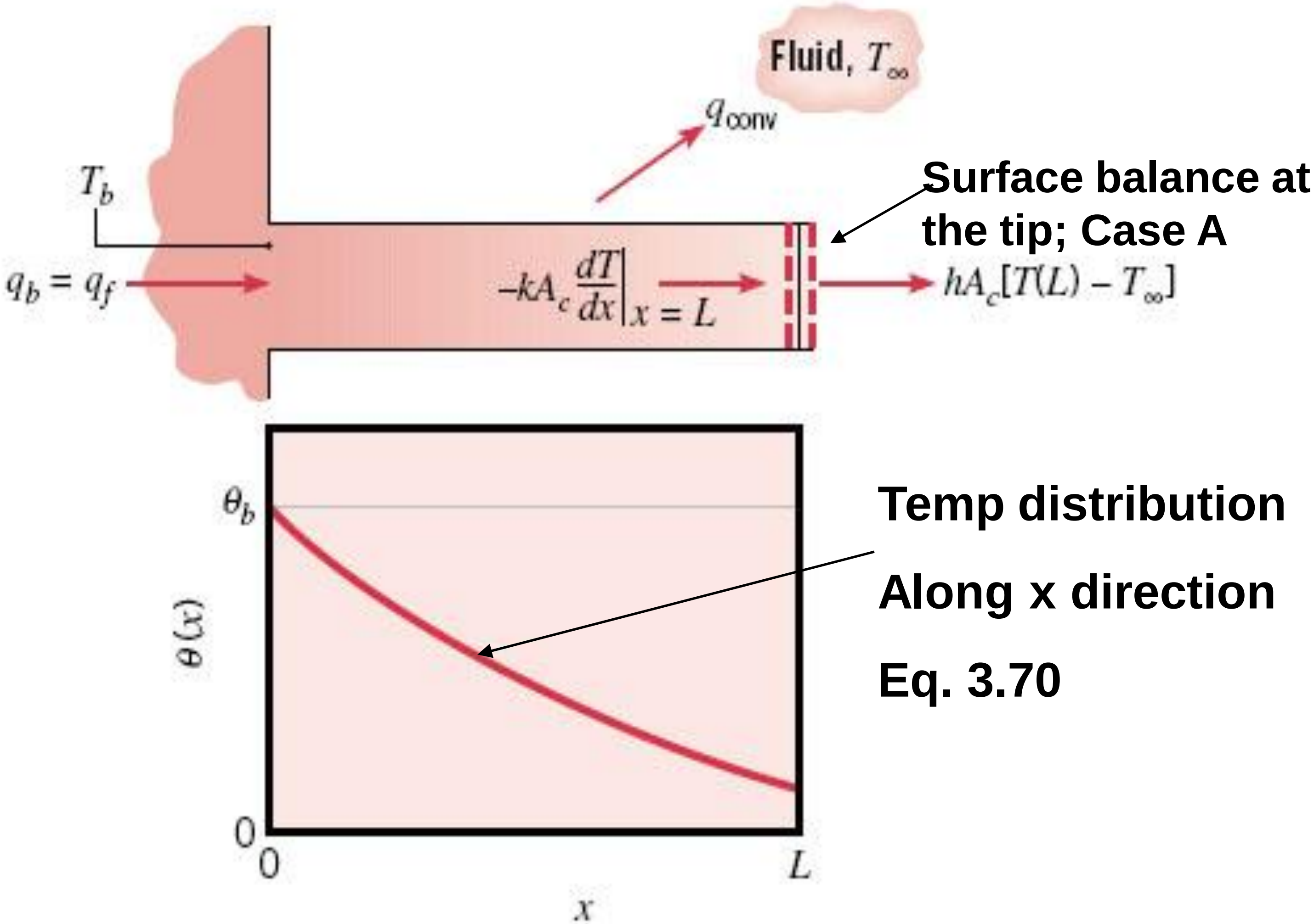


FIGURE 3.17 Conduction and convection in a fin of uniform cross section.

Other cases of tip conditions

TABLE 3.4 Temperature distribution and heat loss for fins of uniform cross section

Case	Tip Condition ($x = L$)	Temperature Distribution θ/θ_b		Fin Heat Transfer Rate q_f	
A	Convection heat transfer: $h\theta(L) = -kd\theta/dx\|_{x=L}$	$\dfrac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$	(3.70)	$M\dfrac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$	(3.72)
B	Adiabatic $d\theta/dx\|_{x=L} = 0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$	(3.75)	$M \tanh mL$	(3.76)
C	Prescribed temperature: $\theta(L) = \theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$	(3.77)	$M\dfrac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$	(3.78)
D	Infinite fin ($L \rightarrow \infty$): $\theta(L) = 0$	e^{-mx}	(3.79)	M	(3.80)

$\theta \equiv T - T_\infty$ $\quad m^2 \equiv hP/kA_c$

$\theta_b = \theta(0) = T_b - T_\infty$ $\quad M \equiv \sqrt{hPkA_c}\,\theta_b$

Fin Performance

- Fin effectiveness, ε

It is defined as the ratio of the fine heat transfer rate to the heat transfer rate that would exist without the fin.

$$\varepsilon = \frac{q_f}{hA_{c,b}\theta_b} \quad (3.81)$$

Where $A_{c,b}$ is the fine cross-sectional area at the base.

- Assume infinite fin L $\rightarrow \infty$, $q_f = M = (hPkA_c)^{1/2}\,\theta_b$

$$\varepsilon = \frac{(hPkA_c)^{1/2}\,\theta_b}{hA_{c,b}\theta_b} = \left(\frac{kP}{hA_c}\right)^{1/2}$$

Notes: 1. effectiveness ↑ by the choice of the materials 'k'

2. effectiveness ↑ by increasing the ratio of perimeter to cross-sectional area

3. max heat rate could be achieved by using very long fins. However, it is not reasonable to use very long fins to achieve near max. heat transfer.

How to obtain a reasonable length?

Since there is no heat transfer from the tip of an infinitely long fin, it more appropriate to compare it with adiabatic tip fin (also no heat loss). Therefore, assume adiabatic tip fin

$$q_f = \sqrt{hPkA_c}\,\theta_b \tanh mL$$

Assume 98% of the max possible heat transfer $q_{f,max} = M = (hPkA_c)^{1/2}\,\theta_b$

$$\therefore \quad 0.98 q_{f,\max} = q_{f,\text{adiabatic}}$$

$$0.98(hPkA_c)^{1/2}\,\theta_b = \sqrt{hPkA_c}\,\theta_b \tanh mL$$

- Hence,

$$mL=2.3 \quad \text{or} \quad L=2.3/m$$

Conclusions

It is more suitable to use fin with L=2.3/m which yield 98% heat transfer rather than to use L >2.3/m or infinite length.

Effectiveness and thermal resistance

$$q_f = \sqrt{hPkA_c}\,\theta_b$$

$$= \frac{\theta_b}{1/\sqrt{hPkA_c}} = \frac{\theta_b}{R_{t,f}}$$

$$\therefore R_{t,f} = \frac{\theta_b}{q_f} = 1/\sqrt{hPkA_c}$$

- Also,

$$q_{f,b}\Big|_{\text{without fin}} = hA_{c,b}\theta_b$$

$$= \frac{\theta_b}{1/hA_{c,b}} = \frac{\theta_b}{R_{t,b}}$$

$$\therefore \varepsilon_f = \frac{q_f}{q_{f,b}} = \frac{\theta_b}{R_{t,f}} \frac{R_{t,b}}{\theta_b} = \frac{R_{t,b}}{R_{t,f}} = \frac{\text{Conduction}}{\text{Convection}}$$

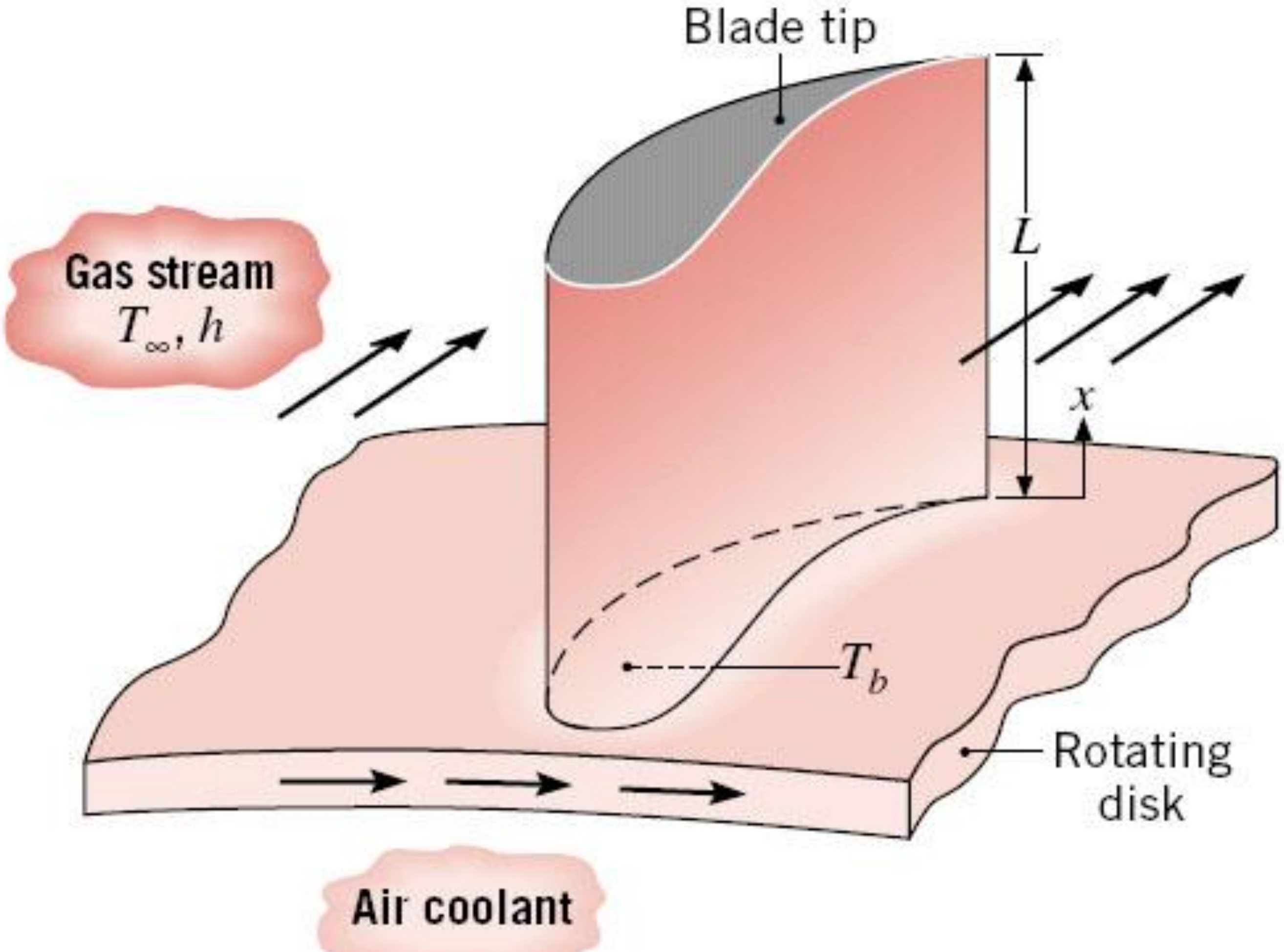
Conclusion

If $\varepsilon > 1$ adding fins enhance heat transfer

$\varepsilon = 1$ adding fins has no effect

$\varepsilon < 1$ adding fins decrease heat transfer

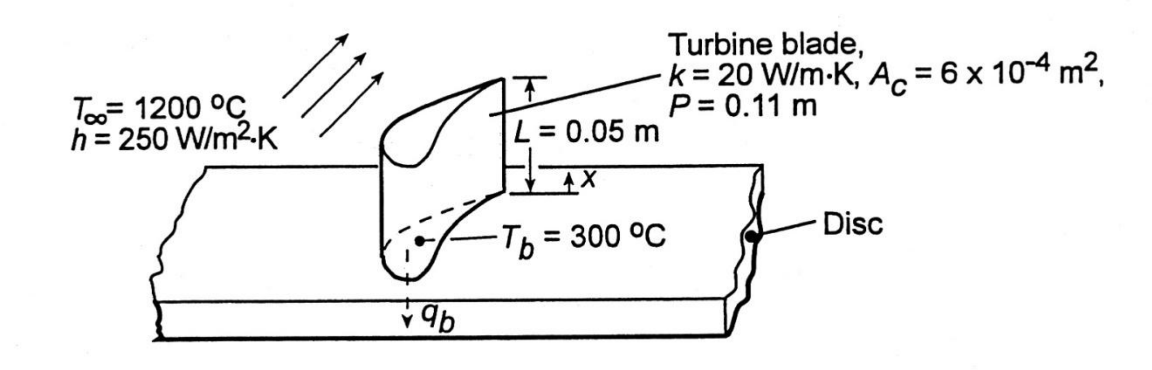
Problem



Blade tip
Gas stream
T_∞, h
L
x
T_b
Rotating
disk
Air coolant

Assessment of cooling scheme for gas turbine blade. Determination of whether blade temperatures are less than the maximum allowable value (1050 °C) for prescribed operating conditions and evaluation of blade cooling rate.

Schematic:



Assumptions: (1) One-dimensional, steady-state conduction in blade, (2) Constant k, (3) Adiabatic blade tip, (4) Negligible radiation.

Analysis: Conditions in the blade are determined by Case B of Table 3.4.

(a) With the maximum temperature existing at x=L, Eq. 3.75 yields

$$\frac{T(L)-T_\infty}{T_b - T_\infty} = \frac{1}{\cosh mL}$$

$$m = \left(hP/kA_c\right)^{1/2} = \left(250\,W/m^2\cdot K \times 0.11\,m/20\,W/m\cdot K \times 6\times 10^{-4}\,m^2\right)^{1/2} = 47.87\ m^{-1}$$

$$mL = 47.87\ m^{-1} \times 0.05\ m = 2.39$$

From Table B.1, $\cosh mL = 5.51$. Hence,

$$T(L) = 1200^\circ C + (300 - 1200)^\circ C/5.51 = 1037^\circ C$$

and, *subject to the assumption of an adiabatic tip*, the operating conditions are acceptable.

(b) With $M = (hPkA_c)^{1/2}\theta_b = \left(250\,W/m^2 \cdot K \times 0.11\,m \times 20\,W/m \cdot K \times 6 \times 10^{-4}\,m^2\right)^{1/2}\left(-900^\circ C\right) = -517\,W$,

Eq. 3.76 and Table B.1 yield

$$q_f = M \tanh mL = -517\,W(0.983) = -508\,W$$

Hence,

$$q_b = -q_f = 508\,W$$

Comments: Radiation losses from the blade surface contribute to reducing the blade temperatures, but what is the effect of assuming an adiabatic tip condition? Calculate the tip temperature allowing for convection from the gas.

Efficiency of Fins, η_f

- Definition:

$$\eta_f = \frac{q_f}{q_{\max}} = \frac{q_f}{hA_f\theta_b}$$

where A_f is the surface area of the fin.

Look! Max heat transfer takes place when the surface temp. of the fin equals the base temperature.

Assume adiabatic tip fin, the previous eq. becomes

$$\eta_f = \frac{M\tanh mL}{hPL\theta_b} = \frac{\sqrt{hPkA_c}\,\theta_b\tanh mL}{hPL\theta_b} = \frac{\tanh mL}{mL}$$

$\eta_f \to$ max as $L \to 0$; $\eta_f \to$ min as $L \to \infty$

Approximation for heat transfer from a convection tip fin

- The heat transfer, q_f, of a convection tip fin; eq. 3.72, can be calculated via using adiabatic tip eq.3.76 by making a correction for the length; L_c=L+(t/2) for a rectangular fin and L_c=L+(D/4) for a pin fin.
- Therefore, with tip convection, the fin heat transfer rate may be approximated as

$$q_f = M \tanh mL_c$$

$$\text{where} \qquad \text{M} = \sqrt{hPkA_c}\,\theta_b$$

$$\text{and} \quad \eta_f = \frac{\tanh mL_c}{mL_c}$$

Notes

1. Errors associated with the approximation are negligible if (ht/k) or (hD/2k) ≤ 0.0625
2. If $w >> t$ for rectangular fin

$\therefore$ $P \approx 2w$

$$mL_c = (\frac{hp}{kA_c})^{1/2} L_c = (\frac{h2w}{kwt})^{1/2} L_c = (\frac{2h}{kt})^{1/2} L_c$$

Introducing a corrected fin profile area, $A_p = L_c t$

$$\therefore \quad mL_c = (\frac{2h}{kt})^{1/2} (\frac{L_c}{L_c})^{1/2} L_c = (\frac{2h}{ktL_c})^{1/2} L_c^{3/2} = (\frac{2h}{kA_p})^{1/2} L_c^{3/2}$$

See Figure 3.18 and figure 3.19

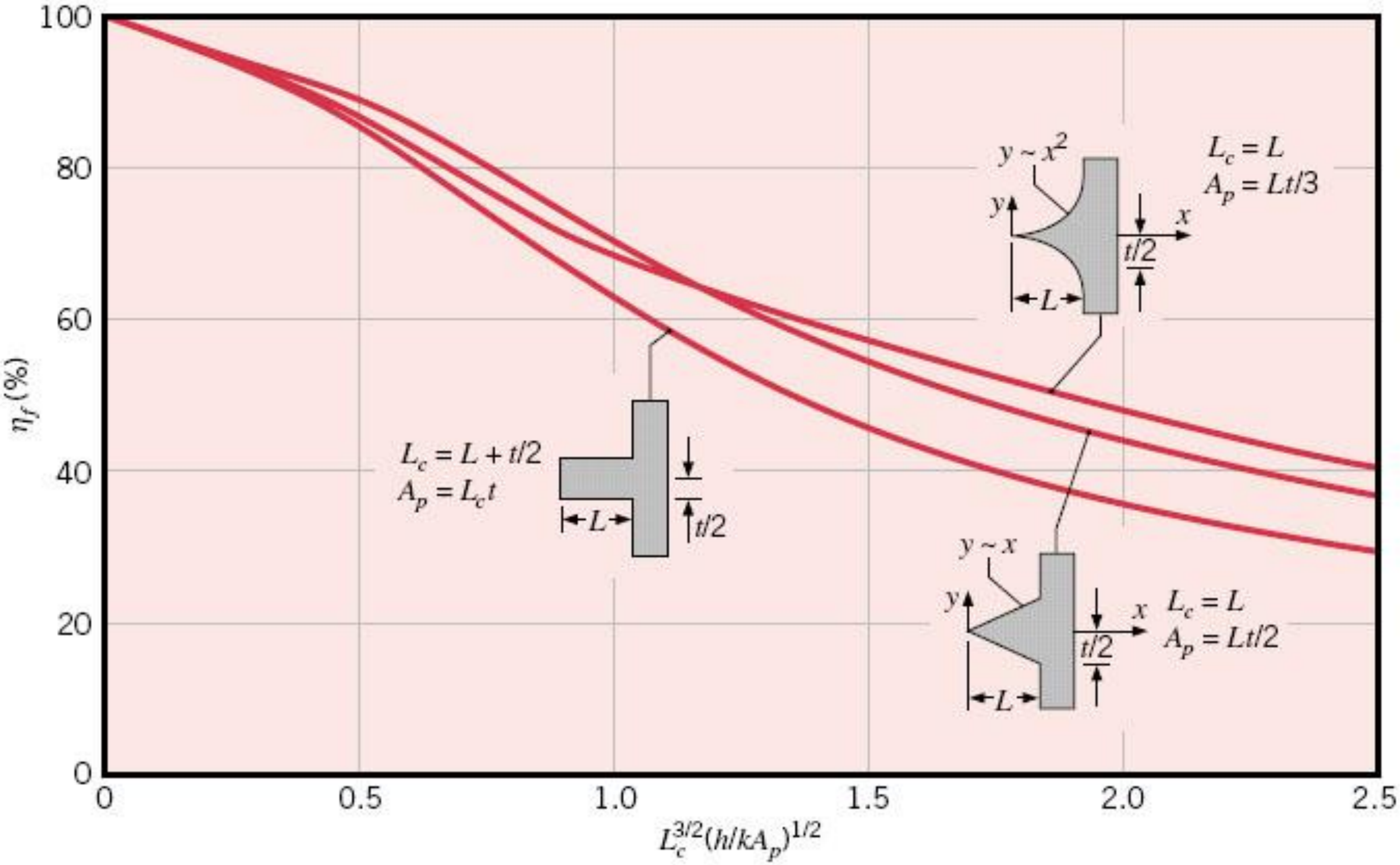


FIGURE 3.18 Efficiency of straight fins (rectangular, triangular, and parabolic profiles).

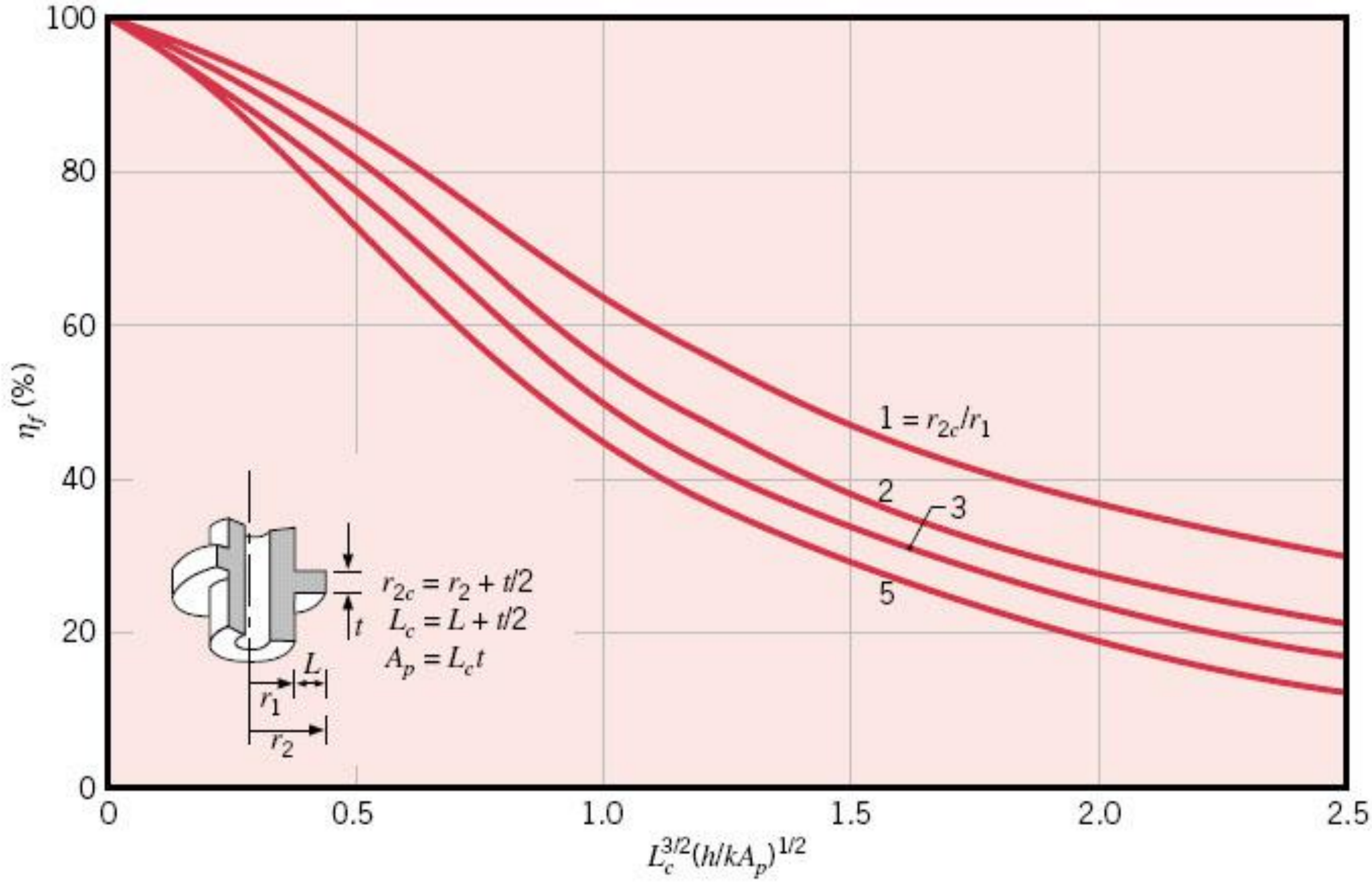


FIGURE 3.19 Efficiency of annular fins of rectangular profile.

summary

$$\therefore \qquad \eta_f = \frac{q_f}{q_{\max}} = \frac{q_f}{hA_f\theta_b}$$

$$q_f = \eta_f q_{\max} = \eta_f A_f \theta_b$$

η_f is obtained from charts or equations

A_f : fin surface area

For example: Pin fin~ $A_f = PL_c = \pi$ D Lc

See next table

TABLE 3.5 Efficiency of common fin shapes

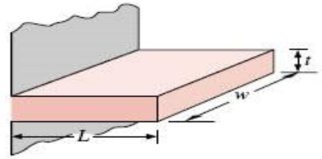
Straight Fins

Rectangular[a]

$A_f = 2wL_c$

$L_c = L + (t/2)$

$A_p = tL$

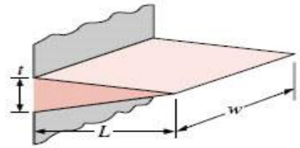


$$\eta_f = \frac{\tanh mL_c}{mL_c} \quad (3.89)$$

Triangular[a]

$A_f = 2w[L^2 + (t/2)^2]^{1/2}$

$A_p = (t/2)L$



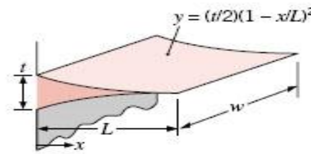
$$\eta_f = \frac{1}{mL}\frac{I_1(2mL)}{I_0(2mL)} \quad (3.93)$$

Parabolic[a]

$A_f = w[C_1 L + (L^2/t)\ln(t/L + C_1)]$

$C_1 = [1 + (t/L)^2]^{1/2}$

$A_p = (t/3)L$



$$\eta_f = \frac{2}{[4(mL)^2 + 1]^{1/2} + 1} \quad (3.94)$$

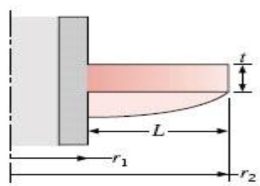
Circular Fin

Rectangular[a]

$A_f = 2\pi(r_{2c}^2 - r_1^2)$

$r_{2c} = r_2 + (t/2)$

$V = \pi(r_2^2 - r_1^2)t$



$$\eta_f = C_2 \frac{K_1(mr_1)I_1(mr_{2c}) - I_1(mr_1)K_1(mr_{2c})}{I_0(mr_1)K_1(mr_{2c}) + K_0(mr_1)I_1(mr_{2c})} \quad (3.91)$$

$$C_2 = \frac{(2r_1/m)}{(r_{2c}^2 - r_1^2)}$$

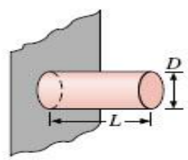
Pin Fins

Rectangular[b]

$A_f = \pi D L_c$

$L_c = L + (D/4)$

$V = (\pi D^2/4)L$

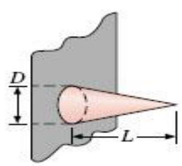


$$\eta_f = \frac{\tanh mL_c}{mL_c} \quad (3.95)$$

Triangular[b]

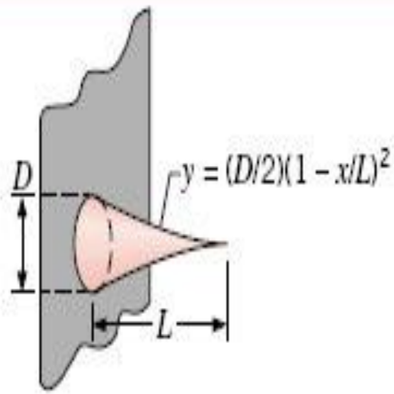
$A_f = \frac{\pi D}{2}[L^2 + (D/2)^2]^{1/2}$

$V = (\pi/12)D^2 L$



$$\eta_f = \frac{2}{mL}\frac{I_2(2mL)}{I_1(2mL)} \quad (3.96)$$

TABLE 3.5 *Continued*

Parabolic[b] $A_f = \frac{\pi L^3}{8D}\{C_3C_4 - \frac{L}{2D}\ln[(2DC_4/L) + C_3]\}$ $C_3 = 1 + 2(D/L)^2$ $C_4 = [1 + (D/L)^2]^{1/2}$ $V = (\pi/20)D^2 L$		$\eta_f = \frac{2}{[4/9(mL)^2 + 1]^{1/2} + 1}$	(3.97)

[a] $m = (2h/kt)^{1/2}$.

[b] $m = (4h/kD)^{1/2}$.