Transient Conduction

- The lumped capacitance Method
- Spatial Effects and the Role of Analytical Solutions

Introduction

$$\boxed{\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

This term accounts the variation of temperature with time for unsteady state problems

Examples

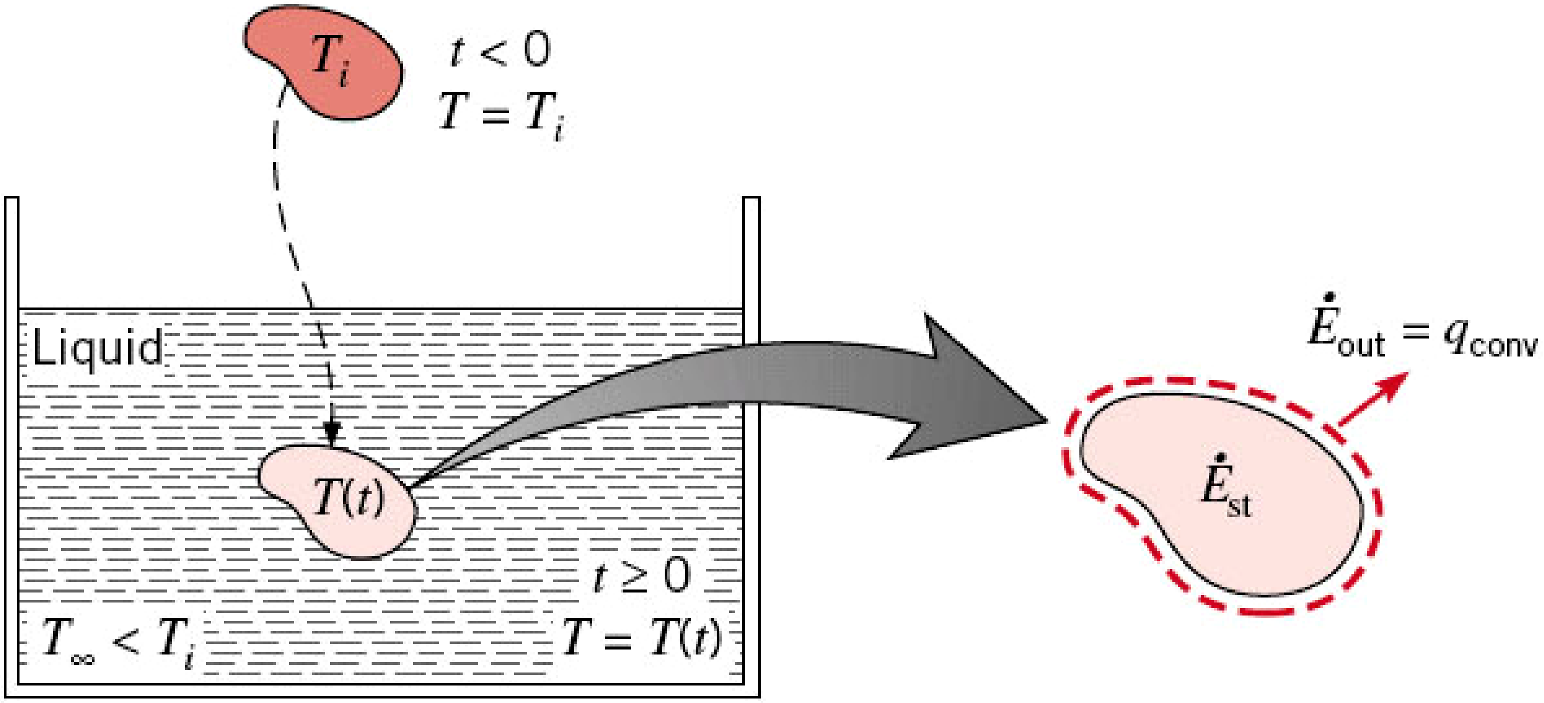
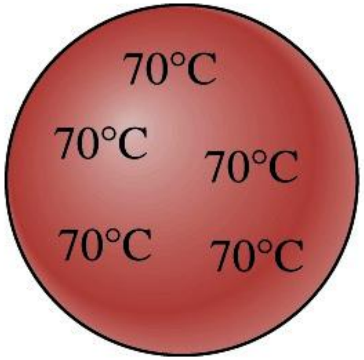


FIGURE 5.1 Cooling of a hot metal forging.

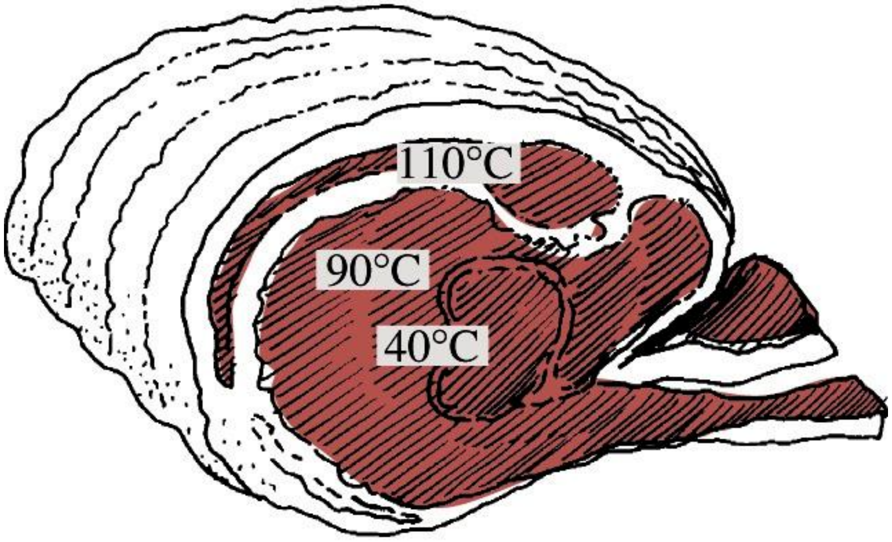
Could be a small copper ball

Examples and comparison



(*a*) Copper ball

$T = f(t)$, $T \neq f(dis)$



(*b*) Roast beef

$T = f(t, dis)$

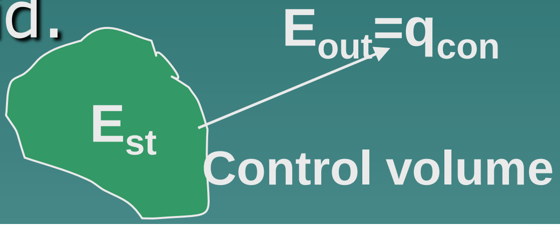
Analysis and general formulation

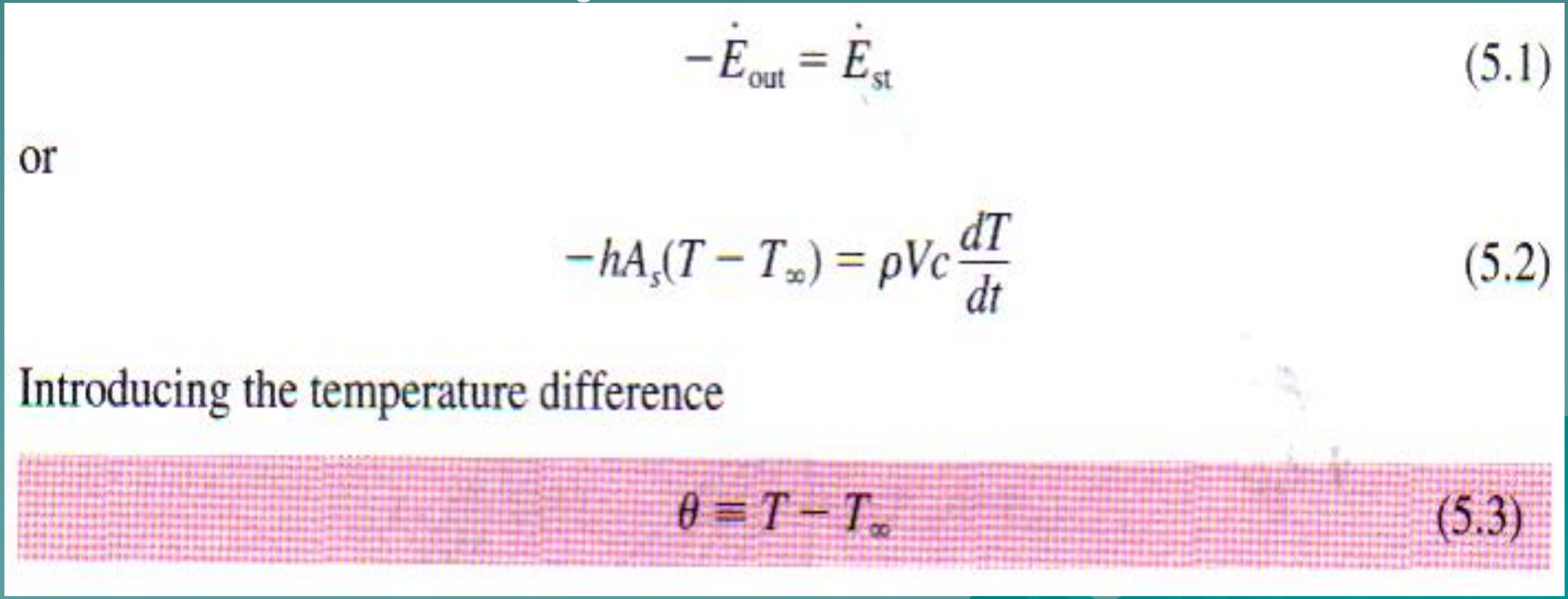
- Basic assumption:

  The lumped capacitance method assumes that the temperature of solid is spatially uniform at any instant of time. This means negligible temp gradients within the solid.

Energy balance:

$$\underset{0}{\cancel{E_{in}}}-E_{out}=E_{st}$$





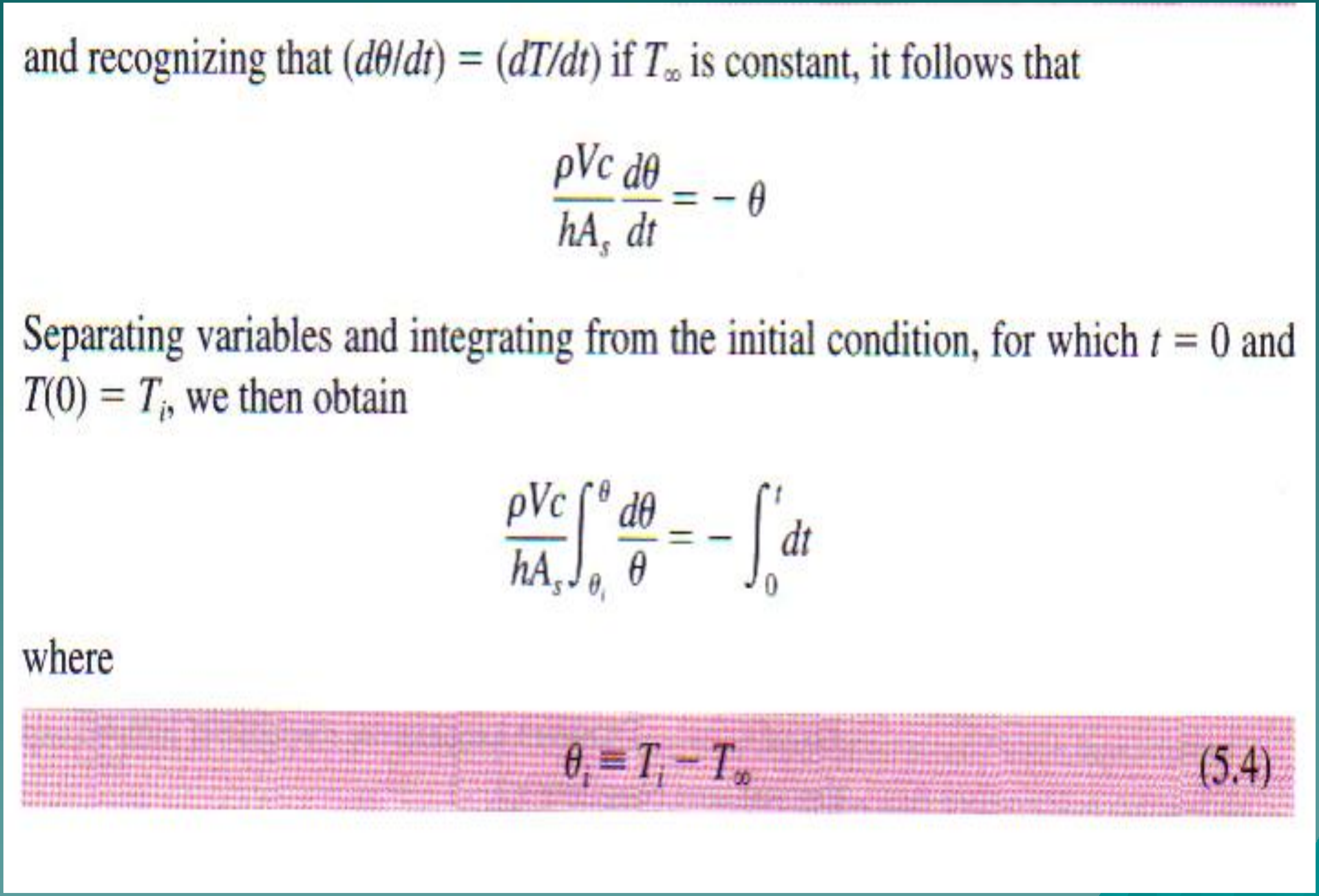
$$-\dot{E}_{out}=\dot{E}_{st} \tag{5.1}$$

or

$$-hA_s(T-T_\infty)=\rho Vc\frac{dT}{dt} \tag{5.2}$$

Introducing the temperature difference

$$\theta = T - T_\infty \tag{5.3}$$



and recognizing that $(d\theta/dt) = (dT/dt)$ if T_∞ is constant, it follows that

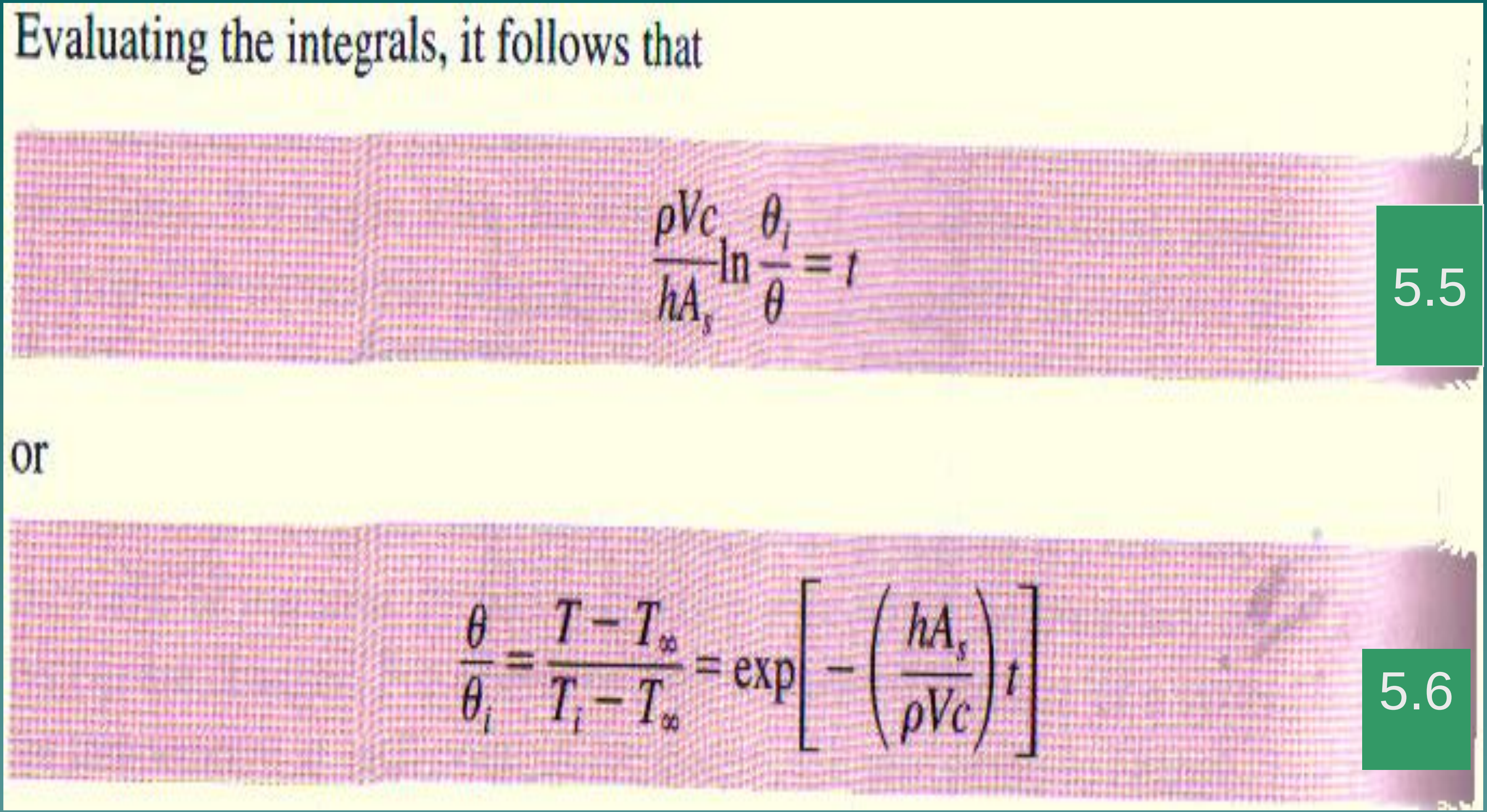
$$\frac{\rho V c}{hA_s}\frac{d\theta}{dt} = -\theta$$

Separating variables and integrating from the initial condition, for which $t = 0$ and $T(0) = T_i$, we then obtain

$$\frac{\rho V c}{hA_s}\int_{\theta_i}^{\theta}\frac{d\theta}{\theta} = -\int_0^t dt$$

where

$$\theta_i \equiv T_i - T_\infty \qquad (5.4)$$



Evaluating the integrals, it follows that

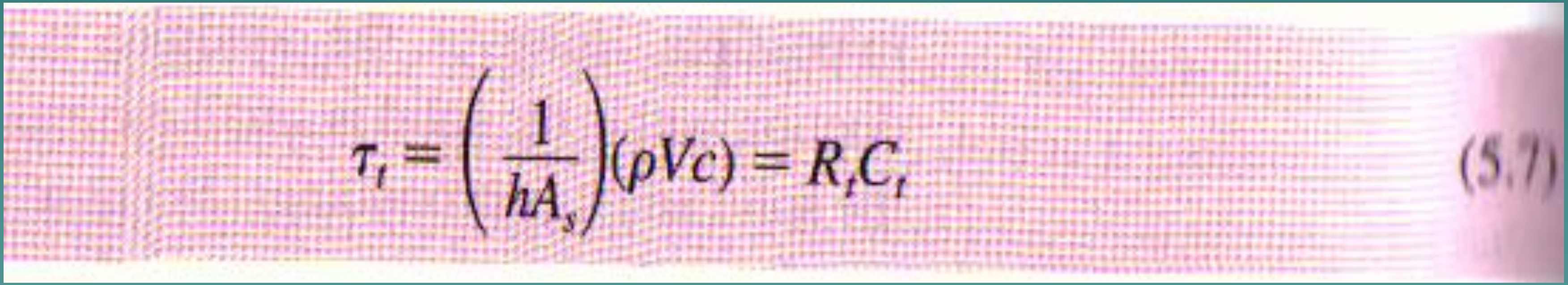
$$\frac{\rho V c}{h A_s}\ln\frac{\theta_i}{\theta} = t \qquad 5.5$$

or

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{h A_s}{\rho V c}\right)t\right] \qquad 5.6$$

Thermal time constant,

$$\tau_t = \rho Vc / hA_s$$



$$\tau_t = \left(\frac{1}{hA_s}\right)(\rho Vc) = R_t C_t \qquad (5.7)$$

Where R_t is the resistance to convection heat transfer and C_t is the lumped thermal capacitance of the solid.

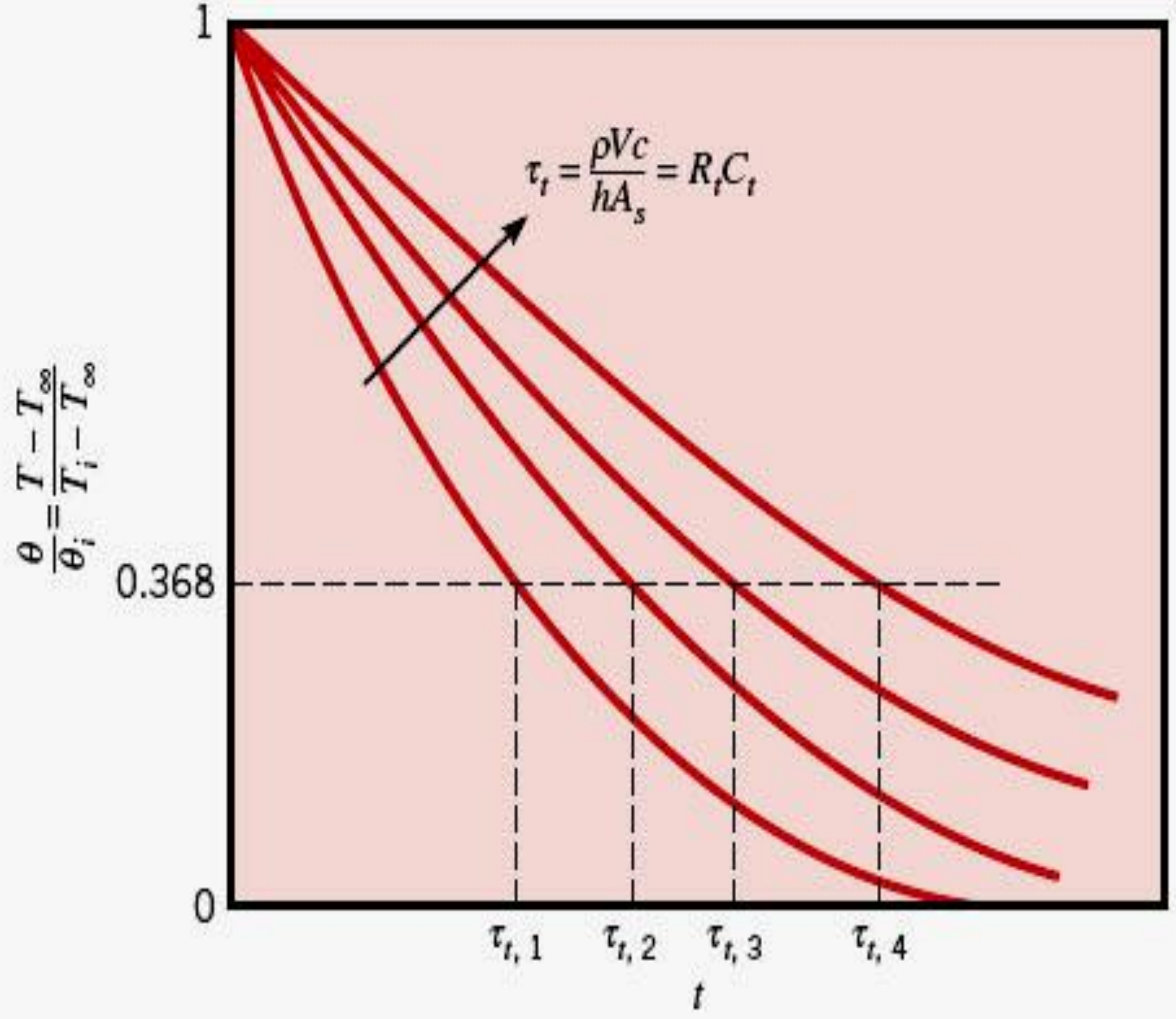


FIGURE 5.2 Transient temperature response of lumped capacitance solids for different thermal time constants τ_t.

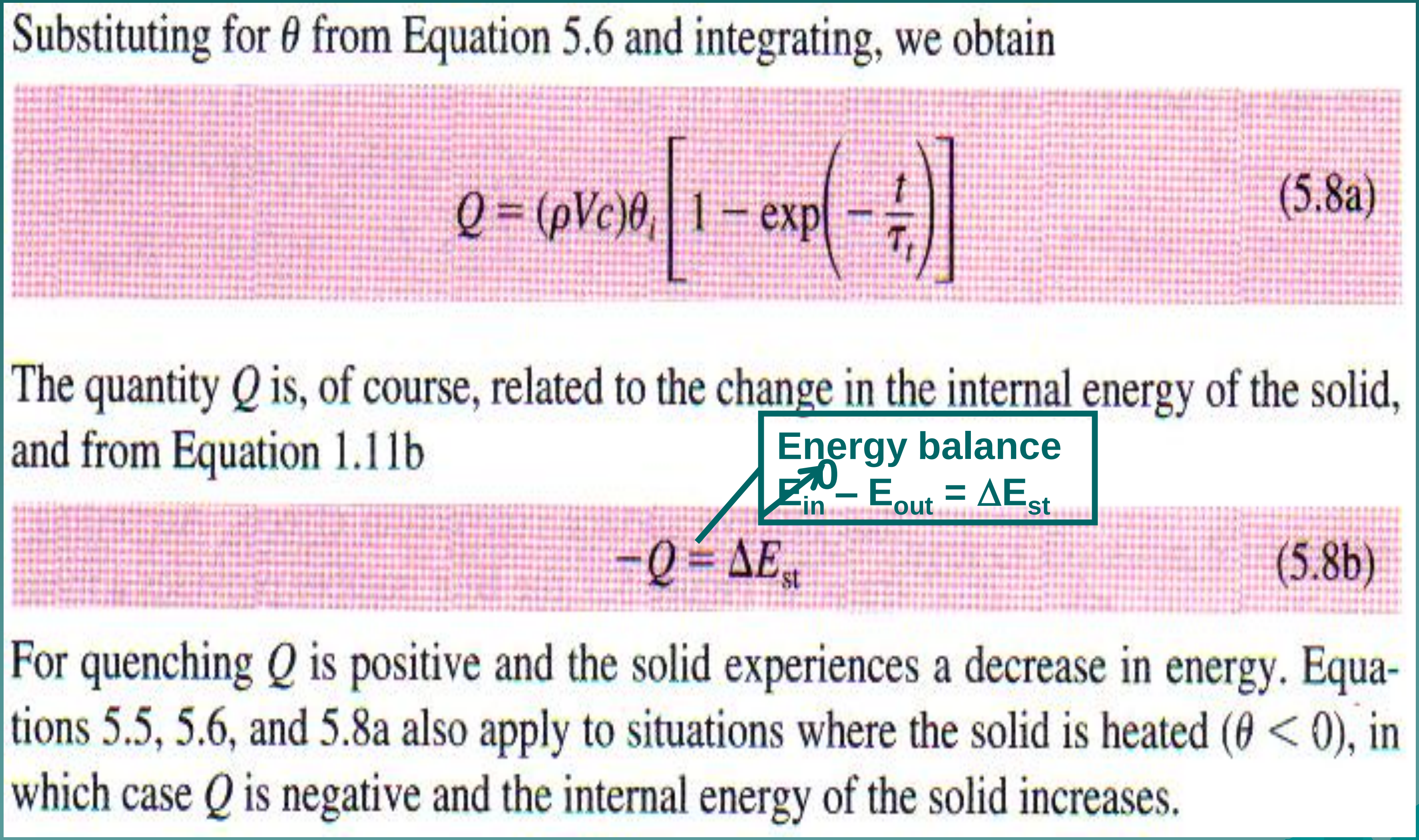
Note:

The temp. of a lumped system approaches the environment temp. as time gets larger.

Total energy transfer Q

- The Total energy transfer Q occurring up to some time t can be obtained from:

$$Q = \int_0^t q \, dt = hA_s \int_0^t \theta \, dt$$



Substituting for θ from Equation 5.6 and integrating, we obtain

$$Q = (\rho V c)\theta_i \left[1 - \exp\left(-\frac{t}{\tau_t}\right)\right] \tag{5.8a}$$

The quantity Q is, of course, related to the change in the internal energy of the solid, and from Equation 1.11b

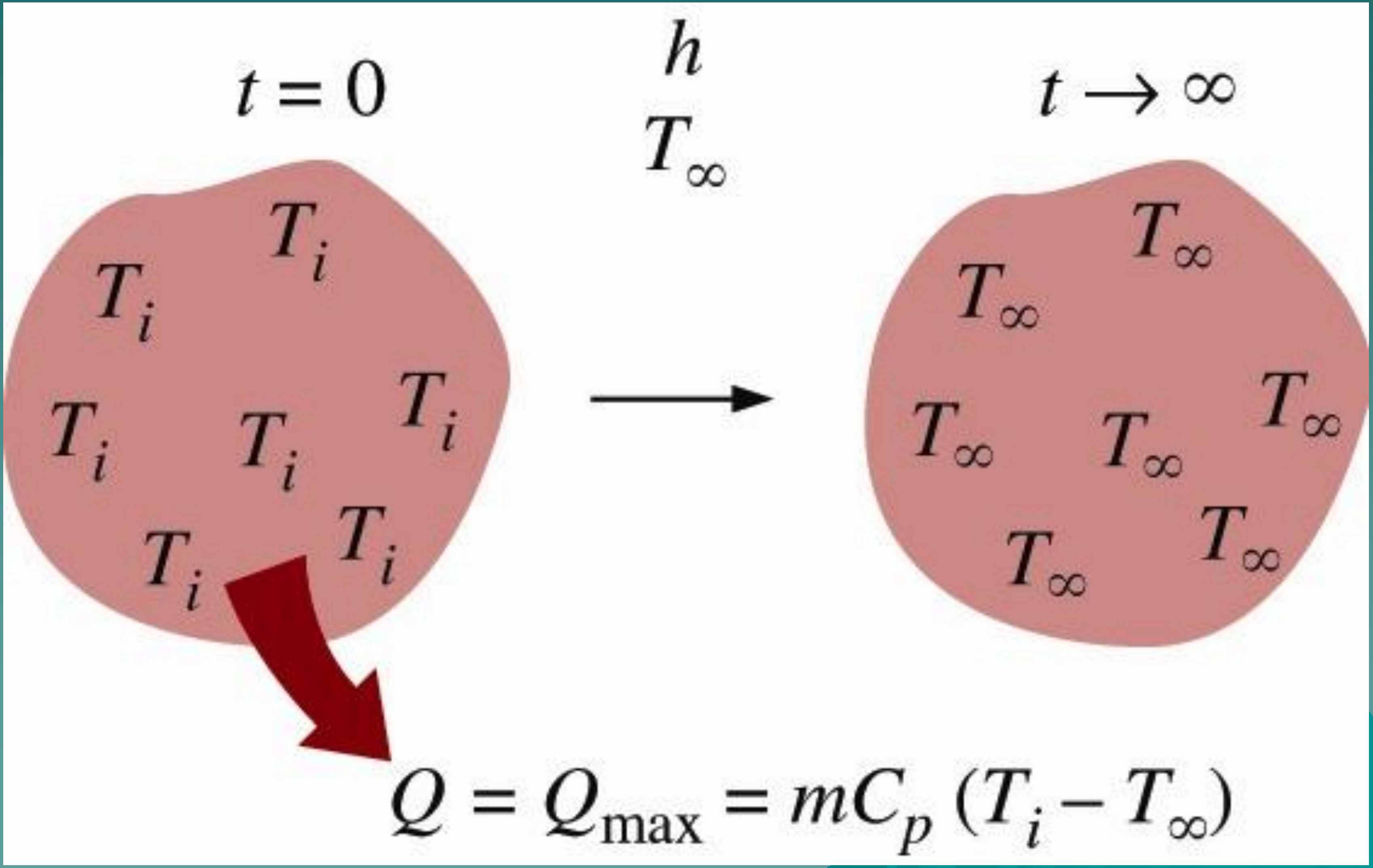
$$-Q = \Delta E_{st} \tag{5.8b}$$

For quenching Q is positive and the solid experiences a decrease in energy. Equations 5.5, 5.6, and 5.8a also apply to situations where the solid is heated ($\theta < 0$), in which case Q is negative and the internal energy of the solid increases.

Energy balance

$\cancelto{0}{E_{in}} - E_{out} = \Delta E_{st}$

Maximum heat quantity



$$Q = Q_{\max} = mC_p\,(T_i - T_\infty)$$

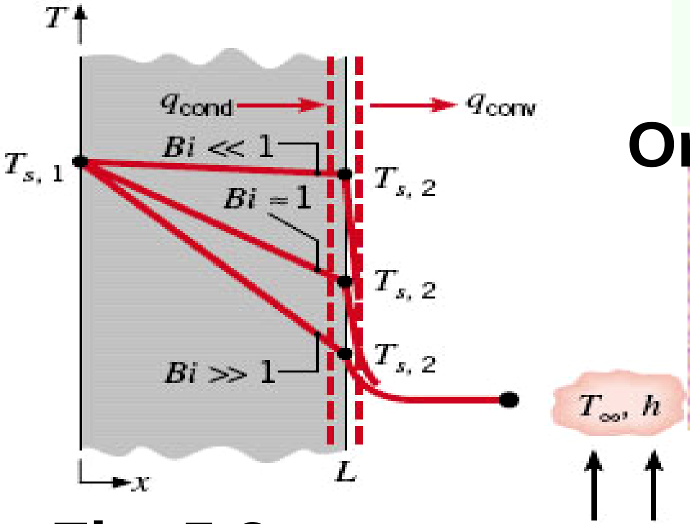
Validity of the Lumped Capacitance Method

To develop a suitable criterion consider steady-state conduction through the plane wall of area A (Figure 5.3). Although we are assuming steady-state conditions, this criterion is readily extended to transient processes. One surface is maintained at a temperature $T_{s,1}$ and the other surface is exposed to a fluid of temperature $T_\infty < T_{s,1}$. The temperature of this surface will be some intermediate value, $T_{s,2}$, for which $T_\infty < T_{s,2} < T_{s,1}$. Hence under steady-state conditions the surface energy balance, Equation 1.12, reduces to

$$\frac{kA}{L}(T_{s,1} - T_{s,2}) = hA(T_{s,2} - T_\infty)$$

Or

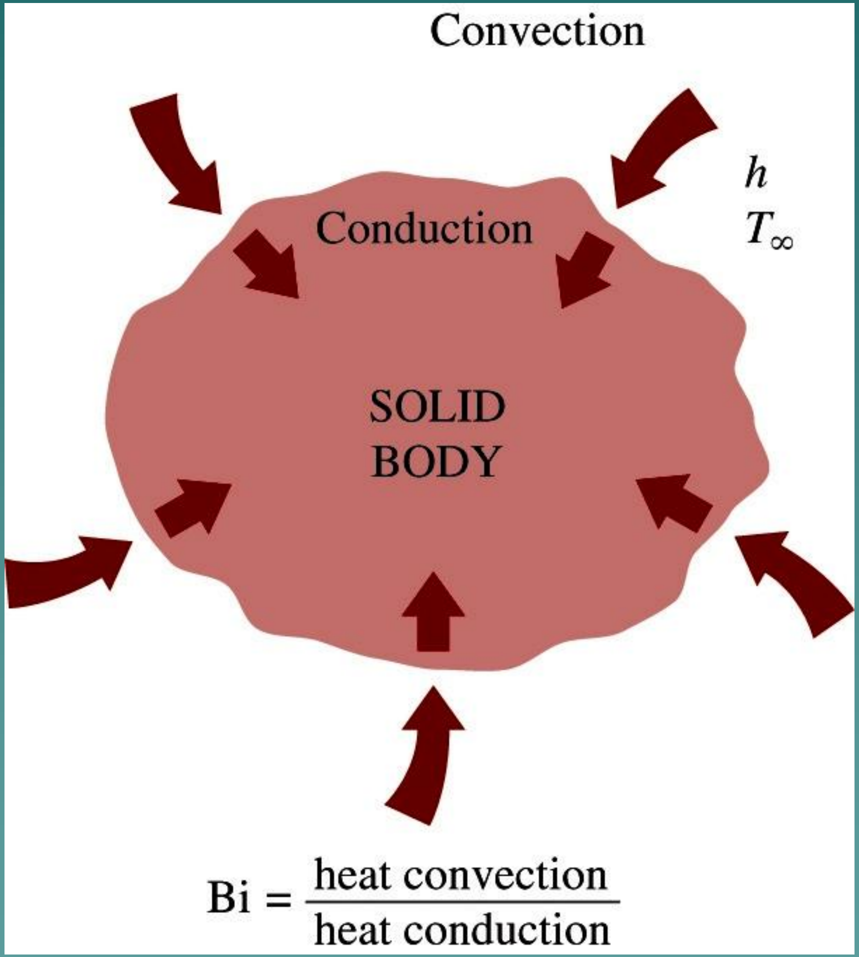
$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_\infty} = \frac{(L/kA)}{(1/hA)} = \frac{R_{\text{cond}}}{R_{\text{conv}}} = \frac{hL}{k} \equiv Bi$$



Effect of Biot number on steady-state temperature distribution in a plane wall with surface convection.

Fig. 5.3

The Biot number can be viewed as the ratio of the convection at the surface to conduction within the body



$$\mathrm{Bi} = \frac{\text{heat convection}}{\text{heat conduction}}$$

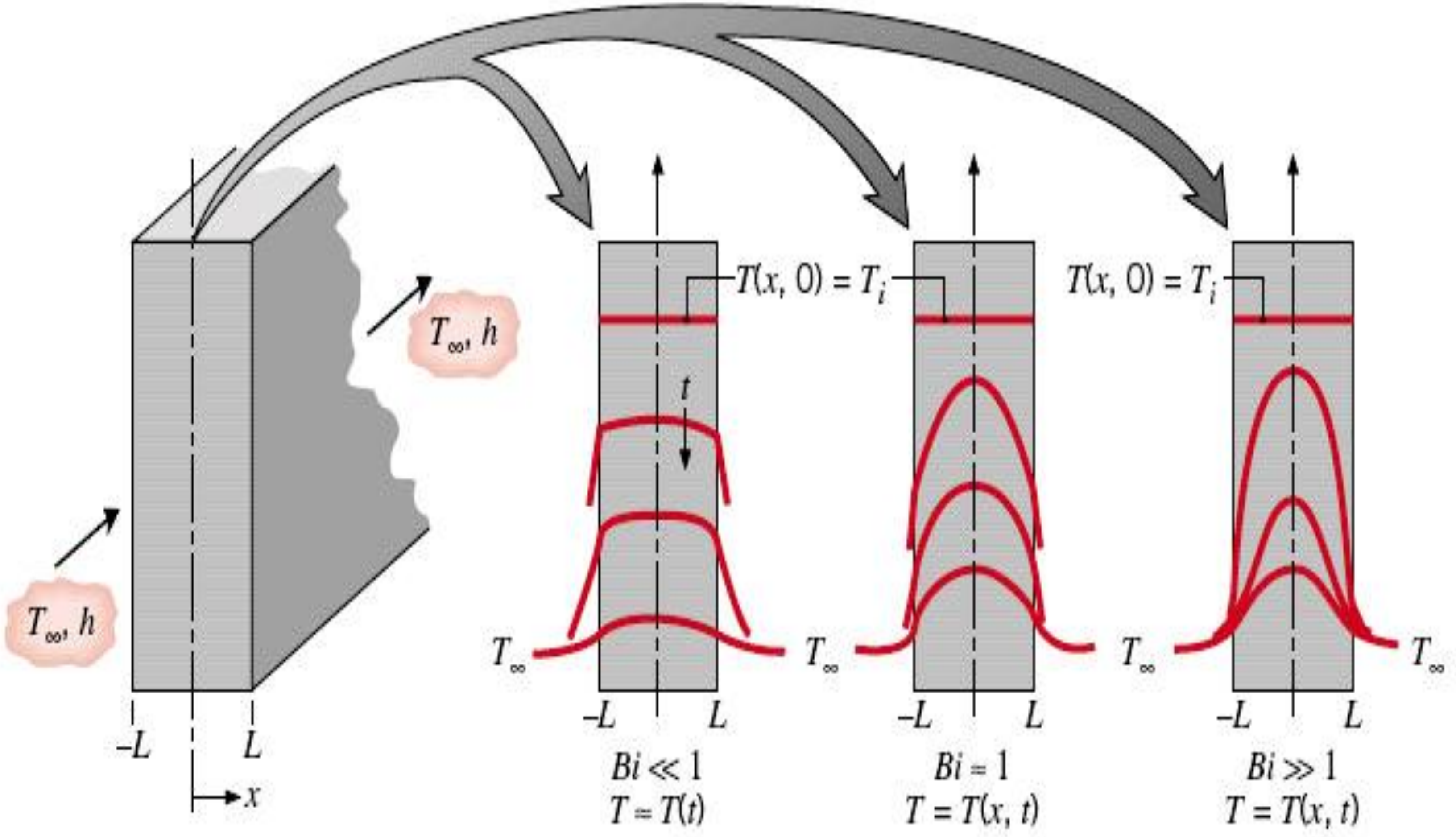


FIGURE 5.4 Transient temperature distributions for different Biot numbers in a plane wall symmetrically cooled by convection.

Conclusion

- If the following condition is satisfied

$$Bi = \frac{hL_c}{k} < 0.1 \qquad 5.10$$

the error associated with using the lumped capacitance method is small.

Where L_c is the characteristic length. It is defined as the ratio of the solid's volume to surface area,

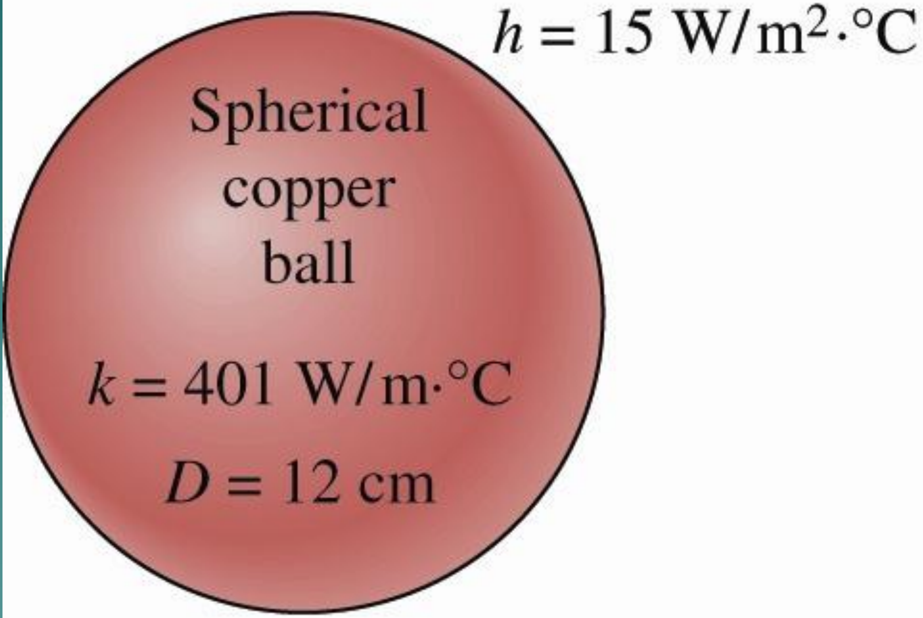
$$L_c = V/A_s$$

Note

For symmetrical heated or cooled plane wall of thickness 2L, $L_c = L$.

For a long cylinder $L_c = r_o/2$ and for a sphere, $L_c = r_o/3$

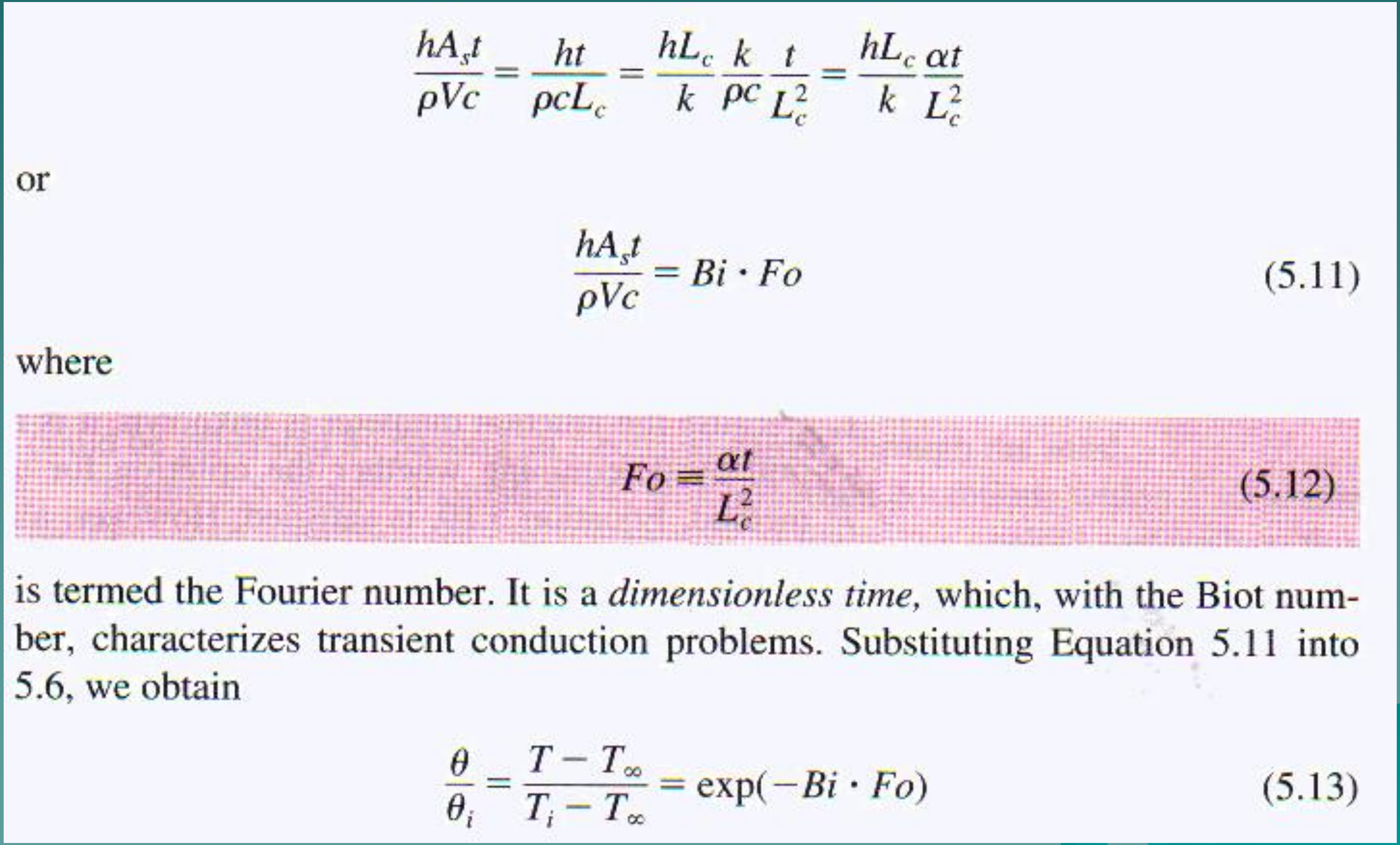
Example: small bodies with high thermal conductivities and low convection coefficients are most likely to satisfy the criterion for lumped system analysis



$$L_c = \frac{V}{A_s} = \frac{\frac{1}{6}\pi D^3}{\pi D^2} = \frac{1}{6}D = 0.02\ \mathrm{m}$$

$$\mathrm{Bi} = \frac{hL_c}{k} = \frac{15 \times 0.02}{401} = 0.00075 < 0.1$$

- Using $L_c = V/A_s$, the exponent of eq. 5.6 may be written as



$$\frac{hA_s t}{\rho V c} = \frac{ht}{\rho c L_c} = \frac{hL_c}{k}\frac{k}{\rho c}\frac{t}{L_c^2} = \frac{hL_c}{k}\frac{\alpha t}{L_c^2}$$

or

$$\frac{hA_s t}{\rho V c} = Bi \cdot Fo \tag{5.11}$$

where

$$Fo \equiv \frac{\alpha t}{L_c^2} \tag{5.12}$$

is termed the Fourier number. It is a *dimensionless time*, which, with the Biot number, characterizes transient conduction problems. Substituting Equation 5.11 into 5.6, we obtain

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp(-Bi \cdot Fo) \tag{5.13}$$

Transient Conduction:
Spatial Effects and the Role of Analytical Solutions

Solution to the Heat Equation for a Plane Wall with Symmetrical Convection Conditions

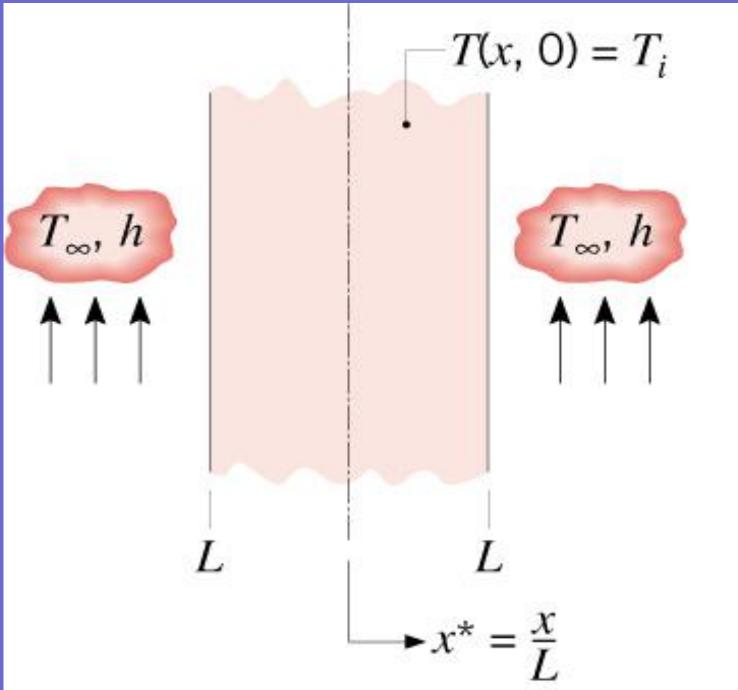
- If the lumped capacitance approximation can not be made, consideration must be given to spatial, as well as temporal, variations in temperature during the transient process.

  - For a plane wall with symmetrical convection conditions and constant properties, the heat equation and initial/boundary conditions are:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (5.26)$$

$$T(x,0) = T_i \qquad (5.27)$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \qquad (5.28)$$



$$-k\frac{\partial T}{\partial x}\bigg|_{x=L} = h\left[T(L,t) - T_{\infty}\right] \qquad (5.29)$$

Existence of seven independent variables:

$$T = T\left(x,t,T_i,T_{\infty},k,\alpha,h\right) \qquad (5.30)$$

How may the functional dependence be simplified?

Non-dimensionalization of Heat Equation and Initial/Boundary Conditions:

Dimensionless temperature difference: $\theta^* \equiv \frac{\theta}{\theta_i} = \frac{T - T_{\infty}}{T_i - T_{\infty}}$

Dimensionless coordinate: $x^* \equiv \frac{x}{L}$

Dimensionless time: $t^* \equiv \frac{\alpha t}{L^2} \equiv Fo$

$Fo \rightarrow$ the Fourier Number

The Biot Number: $Bi \equiv \frac{hL}{k_{solid}}$

$$\theta^* = f\left(x^*, Fo, Bi\right)$$

Substituting the definition of Eq.^{s} 5.31 through 5.33 into Eq.^{s} 5.26 through 5.29 the heat equation becomes

$$\frac{\partial^2 \theta^*}{\partial x^{*2}} = \frac{\partial \theta^*}{\partial Fo} \qquad (5.34)$$

and the initial and boundary conditions become

$$\theta^*(x^*,0) = 1 \qquad (5.35)$$

$$\left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=0} = 0 \qquad (5.36)$$

and

$$\left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=1} = -Bi\,\theta^*(1,t^*) \qquad (5.37)$$

- Exact Solution:

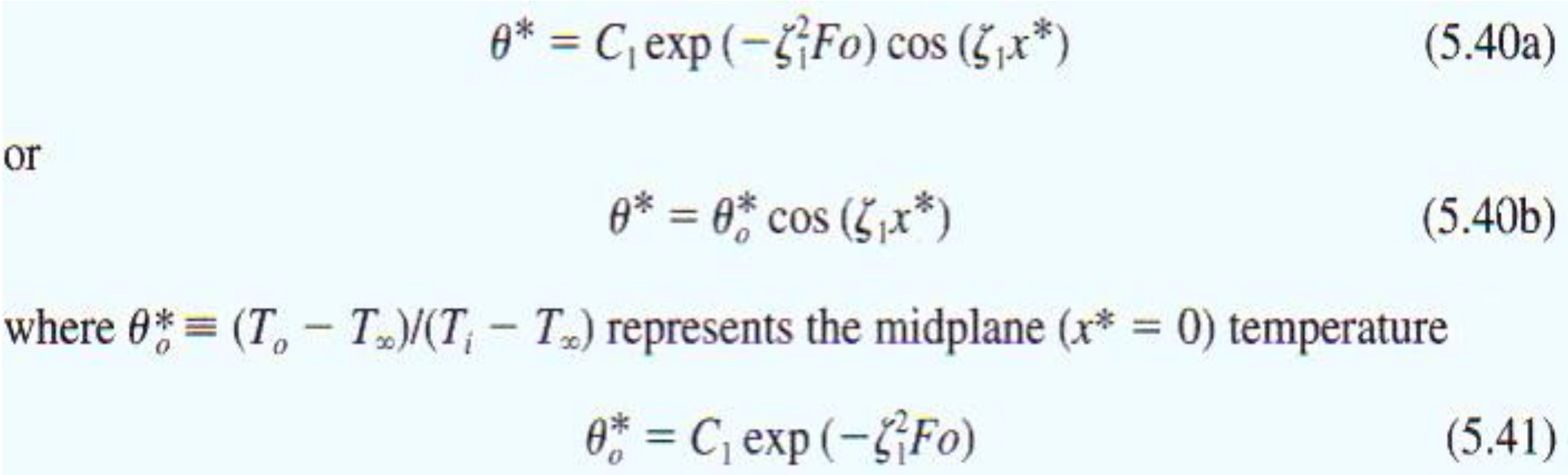
$$\theta^* = \sum_{n=1}^{\infty} C_n \exp\left(-\zeta_n^2 Fo\right)\cos\left(\zeta_n x^*\right) \qquad (5.39a)$$

$$C_n = \frac{4\sin\zeta_n}{2\zeta_n + \sin\left(2\zeta_n\right)} \qquad \zeta_n \tan\zeta_n = Bi \qquad (5.39b,c)$$

See Appendix B.3 for first four roots (eigenvalues $\zeta_1,\ldots,\zeta_4$) of Eq. (5.39c)

Approximate Solution

It was shown that for Fo > 0.2 the infinite series solution eq. 5.39a can be approximated by the 1st term of the series



$$\theta^* = C_1 \exp(-\zeta_1^2 Fo) \cos(\zeta_1 x^*) \qquad (5.40a)$$

or

$$\theta^* = \theta_o^* \cos(\zeta_1 x^*) \qquad (5.40b)$$

where $\theta_o^* \equiv (T_o - T_\infty)/(T_i - T_\infty)$ represents the midplane ($x^* = 0$) temperature

$$\theta_o^* = C_1 \exp(-\zeta_1^2 Fo) \qquad (5.41)$$

C_1 and ξ_1 are given in Table 5.1 for a range of Biot numbers.

Bi=hL/k for Plane wall and hr$_o$/k for the infinite cylinder and sphere

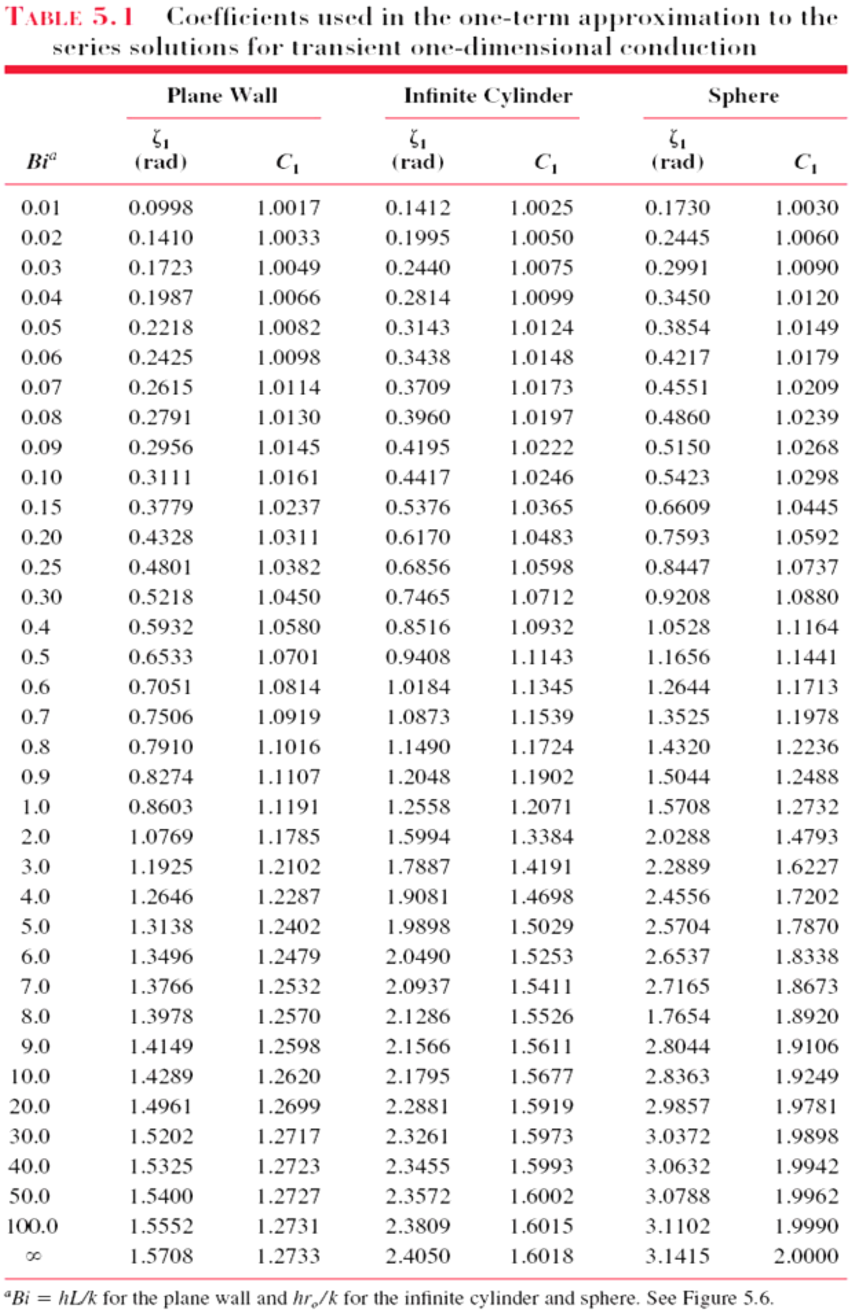
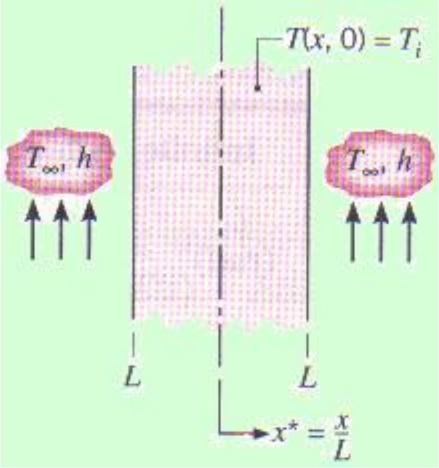


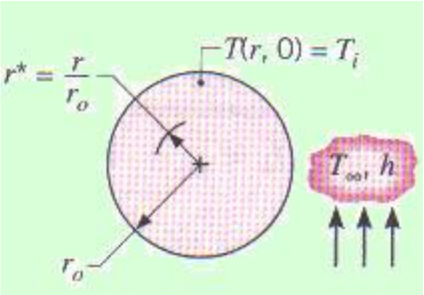
TABLE 5.1 Coefficients used in the one-term approximation to the series solutions for transient one-dimensional conduction

	Plane Wall		Infinite Cylinder		Sphere	
Bi^a	ζ_1 (rad)	C_1	ζ_1 (rad)	C_1	ζ_1 (rad)	C_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.03	0.1723	1.0049	0.2440	1.0075	0.2991	1.0090
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.05	0.2218	1.0082	0.3143	1.0124	0.3854	1.0149
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.07	0.2615	1.0114	0.3709	1.0173	0.4551	1.0209
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.09	0.2956	1.0145	0.4195	1.0222	0.5150	1.0268
0.10	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.15	0.3779	1.0237	0.5376	1.0365	0.6609	1.0445
0.20	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.25	0.4801	1.0382	0.6856	1.0598	0.8447	1.0737
0.30	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0932	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0919	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5994	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2402	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	1.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2881	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2733	2.4050	1.6018	3.1415	2.0000

[a] $Bi = hL/k$ for the plane wall and hr_o/k for the infinite cylinder and sphere. See Figure 5.6.



Plane wall



Infinite Cylinder Or sphere

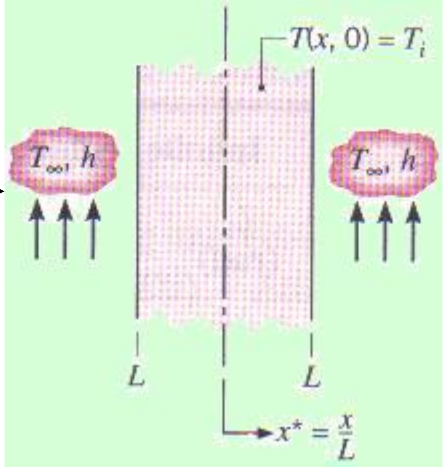
1-D system with an initial uniform temp subjected to sudden convection condition

Total Energy transfer Q left or entered the wall up to any time t in transient process

- Energy equation cab be applied over a time interval t=0 to any time t>0

$$\underset{\text{Zero}}{E_{in}} - \underset{=Q}{E_{out}} = \Delta E_{st} \qquad (5.42)$$

Look!



$\Delta E_{st} = E(t) - E(0)$

Eq. 5.42 becomes

$Q = -[E(t) - E(0)]$

Or

$$Q = -\int \rho C\, [T(x,t) - T_i]\, dV \qquad (5.43)$$

It is convenient to nondimensionalize the result of integration by adopting this quantity

$$Q_0 = \rho C\, V\, (T_i - T_\infty) \qquad (5.44)$$

which may be interpreted as the initial internal energy of the wall relative to the fluid temperature. It is also the *maximum* amount of energy transfer that could occur if the process were continued to time $t = \infty$. Hence, assuming constant properties, the ratio of the total energy transferred from the wall over the time interval t to the maximum possible transfer is

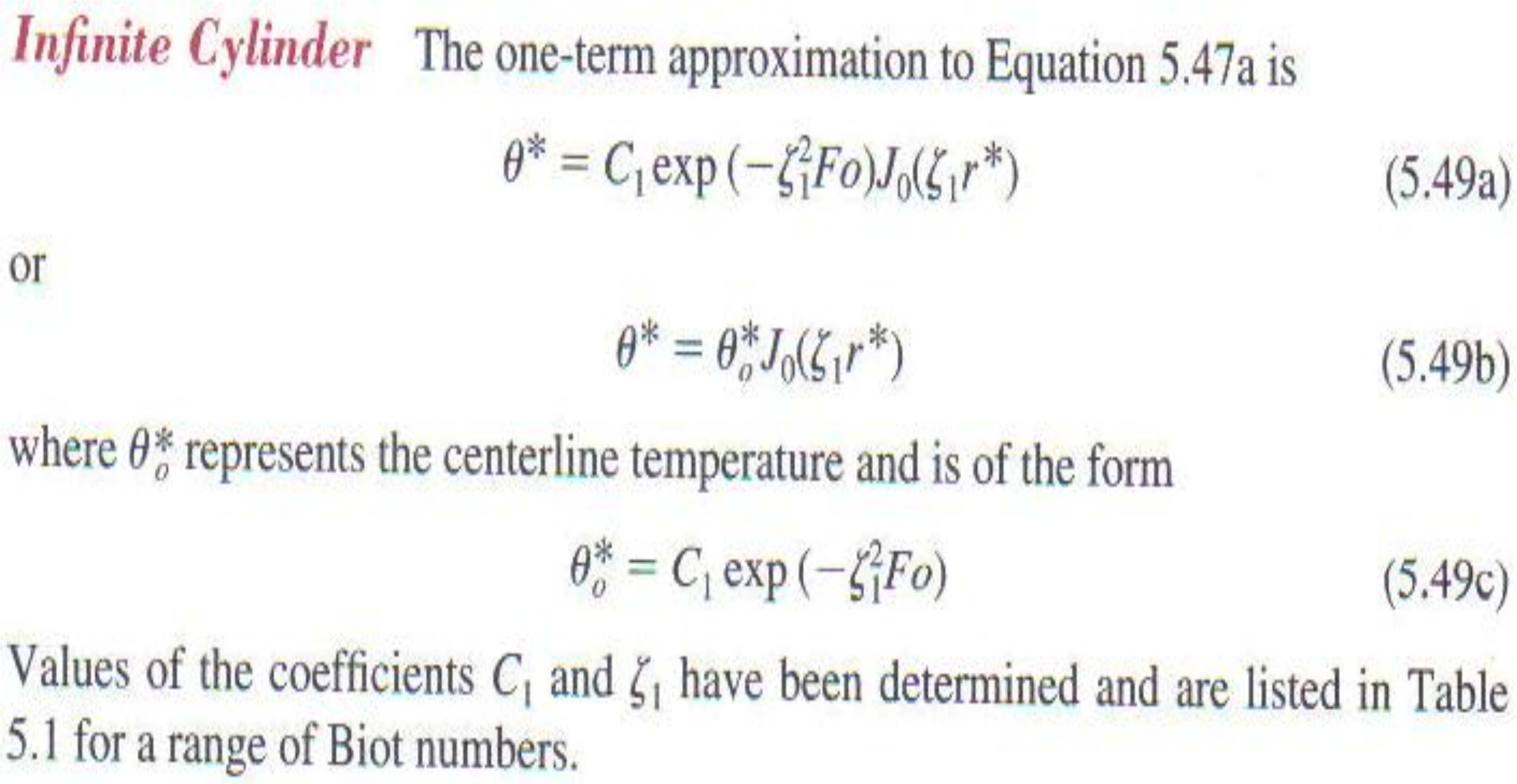
$$\frac{Q}{Q_o} = \int \frac{-[T(x,t) - T_i]}{T_i - T_\infty} \frac{dV}{V} = \frac{1}{V} \int (1 - \theta^*) dV \tag{5.45}$$

Employing the approximate form of the temperature distribution for the plane wall, Equation 5.40b, the integration prescribed by Equation 5.45 can be performed to obtain

$$\frac{Q}{Q_o} = 1 - \frac{\sin \zeta_1}{\zeta_1} \theta_o^* \tag{5.46}$$

where θ_o^* can be determined from Equation 5.41, using Table 5.1 for values of the coefficients C_1 and ζ_1.

Approximate Solution
infinite cylinder



Infinite Cylinder The one-term approximation to Equation 5.47a is

$$\theta^* = C_1 \exp(-\zeta_1^2 Fo) J_0(\zeta_1 r^*) \tag{5.49a}$$

or

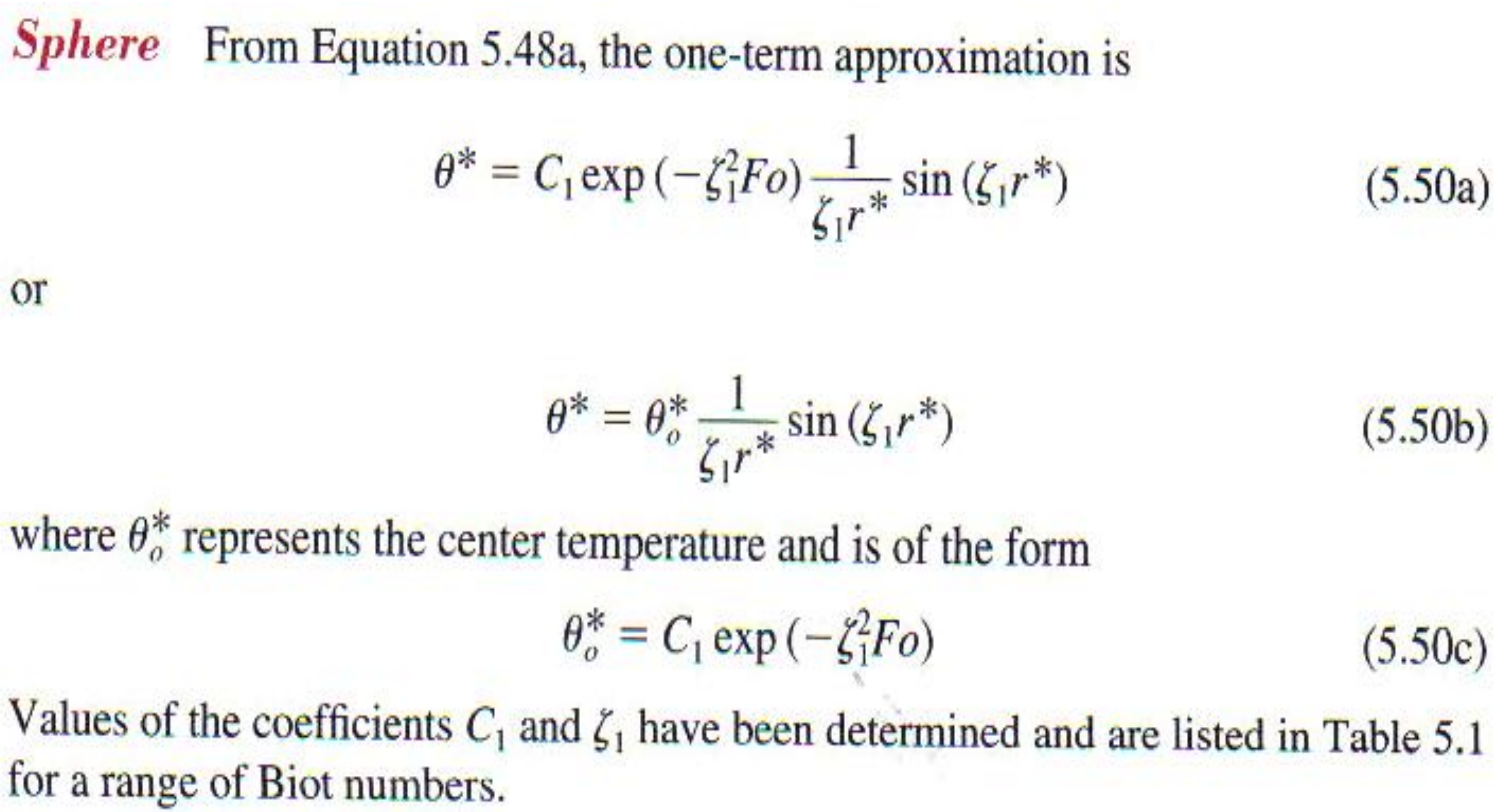
$$\theta^* = \theta_o^* J_0(\zeta_1 r^*) \tag{5.49b}$$

where θ_o^* represents the centerline temperature and is of the form

$$\theta_o^* = C_1 \exp(-\zeta_1^2 Fo) \tag{5.49c}$$

Values of the coefficients C_1 and ζ_1 have been determined and are listed in Table 5.1 for a range of Biot numbers.

Approximate Solution
Sphere



Sphere From Equation 5.48a, the one-term approximation is

$$\theta^* = C_1 \exp(-\zeta_1^2 Fo) \frac{1}{\zeta_1 r^*} \sin(\zeta_1 r^*) \tag{5.50a}$$

or

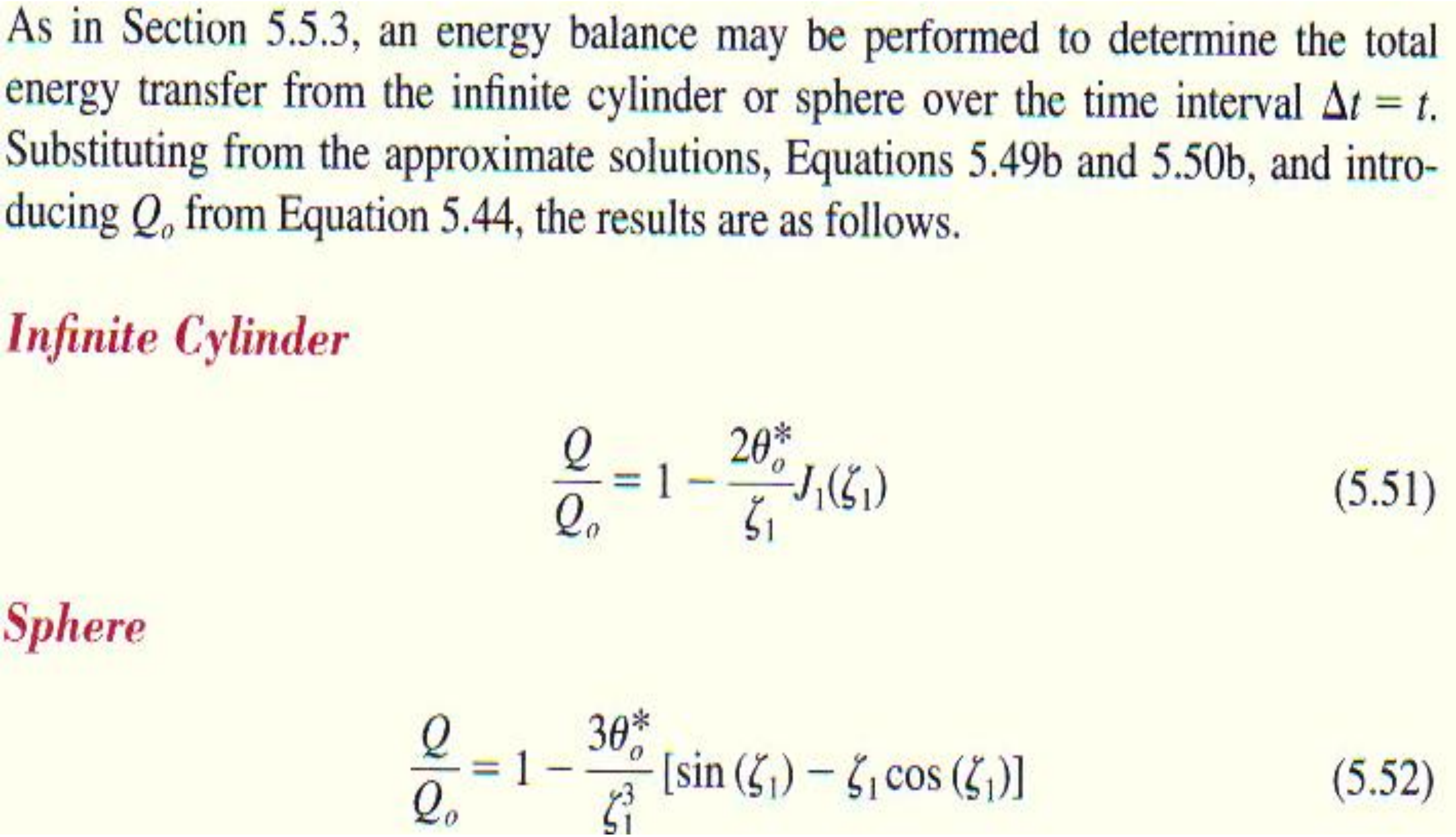
$$\theta^* = \theta_o^* \frac{1}{\zeta_1 r^*} \sin(\zeta_1 r^*) \tag{5.50b}$$

where θ_o^* represents the center temperature and is of the form

$$\theta_o^* = C_1 \exp(-\zeta_1^2 Fo) \tag{5.50c}$$

Values of the coefficients C_1 and ζ_1 have been determined and are listed in Table 5.1 for a range of Biot numbers.

Total Heat transfer



As in Section 5.5.3, an energy balance may be performed to determine the total energy transfer from the infinite cylinder or sphere over the time interval $\Delta t = t$. Substituting from the approximate solutions, Equations 5.49b and 5.50b, and introducing Q_o from Equation 5.44, the results are as follows.

Infinite Cylinder

$$\frac{Q}{Q_o} = 1 - \frac{2\theta_o^*}{\zeta_1} J_1(\zeta_1) \tag{5.51}$$

Sphere

$$\frac{Q}{Q_o} = 1 - \frac{3\theta_o^*}{\zeta_1^3} [\sin(\zeta_1) - \zeta_1 \cos(\zeta_1)] \tag{5.52}$$