PART I

SOLVING LINEAR EQUATION SYSTEMS

$-\frac{1}{2}$ 1st ; (+) to eq #2

$-\frac{3}{2}$ (+) #(3)

$-\frac{1}{2}$ (+) #(4)

(1) INTRODUCTION

You have undoubtedly seen linear equations before. Here, for example, are 4 equations in 4 unknowns:

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 + 5x_4 &= 3 \\ x_1 - x_2 + x_3 + \boxed{2x_4} &= 1 \\ 3x_1 + 2x_2 + 2x_3 - x_4 &= 0 \\ x_1 + x_2 - 3x_3 - x_4 &= 0 \end{aligned}$$

$-\frac{1}{2}$ substract $\frac{1}{2}$ 2nd from 1st

$\frac{3}{2}$ 2nd from 1st

$\frac{1}{2}$ 2nd

The idea is to determine values of the variables x_1, x_2, x_3, x_4 that will satisfy all 4 equations.

More generally, we will write a "system" of m linear equations in n variables as

equation = row column = variable

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

j variables

i equations a_{ij}

$\{a_{ij}\}$ $i = 1, n$ $j = 1, n$

$a_{ij} \equiv$ coef. of the variables x_j in all equations

The values a_{ij} are the coefficients of the variables x_j, and the b_i are the constants of "right hand side" values

We will concentrate on the case $m = n$ case but many of our method will be applicable in the more general case

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}$$

② SOLVING LINEAR SYSTEMS: AN EXAMPLE

You probably have some idea how to go about "solving" th preceding example to get the desired values of the variable

As a start, take a closer look at this solution process:

Elimination (substitution)

a: Using the first equation, you can solve for x_1 in terms of the other variables:

$x_1 = \frac{3}{2}x_2 - x_3 - \frac{5}{2}x_4 + \frac{3}{2} = 1.5 + 1.5x_2 - x_3 - 2.5x_4$

$$x_1 = 1.5 + 1.5x_2 - x_3 - 2.5x_4$$

eliminate x_1

Then you can substitute this expression for x_1 in the other three equations, to get

$$\begin{aligned} +0.5x_2 + 0x_3 - 0.5x_4 &= -0.5 \\ 6.5x_2 - x_3 - 8.5x_4 &= -4.5 \\ 2.5x_2 - 4x_3 - 3.5x_4 &= -1.5 \end{aligned}$$

$\frac{6.5}{.5} = 13$

$\frac{2.5}{.5} = 5$

Using the first remaining equation, you can solve for x_2:

$$x_2 = -1 - 0x_3 + x_4$$

Then you can substitute this expression for x_2 in the other two remaining equations:

-4 $\quad 1$

$$\begin{aligned} -x_3 - 2x_4 &= 2 \\ -4x_3 - x_4 &= 1 \end{aligned}$$

$\frac{-4}{-1} = +4$

$+4x_3 + 8x_4 = -8$

$-4x_3$

Finally, you can use the first of these to solve for x_3:

$$x_3 = -2 - 2x_4$$

and can substitute this expression for x_3 in the second, to g

$$7x_4 = -7$$

Obviously this one equation is solved by $x_4 = -1$.

Now you've finished all the hard work,
and you can easily "substitute back" to get the solution:

$$x_4 = -1 \Rightarrow x_3 = -2 - 2x_4 = 0$$

$$\Rightarrow x_2 = -1 + x_4 = -2$$

$$\Rightarrow x_1 = 1.5 + 1.5x_2 - x_3 - 2.5x_4 = 1$$

Thus $x_1 = 1$, $x_2 = -2$, $x_3 = 0$, $x_4 = -1$ satisfies the equations.

b: There is another way to view the above operations.
It relies on the following observation:

> Starting with any system of equations,
> you can derive an equivalent system ...
>
> by subtracting any multiple of any one equation from any other equation.

By an "equivalent" system of equations, *same sets of solution*
we mean a system that has exactly the same solutions
as the original system that you started with.

Consider what happens if you start with the orginal system
from ①, and you make the following subtractions:

> subtract 0.5 times 1st equation from 2nd equation
> subtract 1.5 times 1st equation from 3rd equation
> subtract 0.5 times 1st equation from 4th equation

The resulting equivalent system is then as follows:

$$\begin{aligned}
2x_1 - 3x_2 + 2x_3 + 5x_4 &= 3 \\
+0.5x_2 \qquad\quad - 0.5x_4 &= -0.5 \\
6.5x_2 - x_3 - 8.5x_4 &= -4.5 \\
2.5x_2 - 4x_3 - 3.5x_4 &= -1.5
\end{aligned}$$

(2) The first equaiion is unchanged. But in the othe equations, subtracting off a multiple of the irst made the coefficient of x_1 exactly zero.

Notice that the last three equations are n w exac same as what you arrived at in (a) before, when for x_1 and substituted.

To continue, suppose you subtract just enough of th equation from the 3rd and 4th equation so as to ma coefficients of x_2 exactly zero. To do so, you must

subtract 13 times the 2nd equation from t 3rd

6.5/.5

subtract 5 times the 2nd equation from the 4th

2.5/0.5

and the resulting equivalent system is as follows:

$$\begin{aligned}
2x_1 - 3x_2 + 2x_3 + 5x_4 &= 3 \\
0.5x_2 \qquad - 0.5x_4 &= -0.5 \\
-x_3 - 2x_4 &= 2 \\
-4x_3 - x_4 &= 1
\end{aligned}$$

$\frac{-4}{-1} = +4$

The last two equations are now exactly the same as wh arrived at in (a) before, when you solved for x_2 and s

To finish the procedure, subtract just enough of the 3 equation from the 4th equation to make the coefficien x_3 exactly zero. That is,

subtract 4 times the 3rd equation from the 4th equat

and you end up with the following equivalent system:

$$\boxed{\begin{aligned}
2x_1 - 3x_2 + 2x_3 + 5x_4 &= 3 \\
0.5x_2 \qquad - 0.5x_4 &= -0.5 \\
-x_3 - 2x_4 &= 2 \\
7x_4 &= -7
\end{aligned}}$$

you end up

substitution Back

GAUSSIAN ELIMINATION

Now you have all the same equations that you derived in (a) — and you can once again "substitute back" to get the solution.

c: As you have seen, the approaches employed in (a) and (b) lead to essentially the same series of calculations.

Thus this section has described essentially just one method for solving linear systems.

This method is known as GAUSSIAN ELIMINATION.

$$\begin{cases} 2x_1 - 3x_2 + 2x_3 + 5x_4 = 3 \\ x_1 - x_2 + x_3 + 2x_4 = 1 \\ 3x_1 + 2x_2 + 2x_3 - x_4 = 0 \\ x_1 + x_2 - 3x_3 - x_4 = 0 \end{cases}$$

→ Determine values of x_1, x_2, x_3, x_4

$$\boxed{x_1 = 1.5 + 1.5x_2 - x_3 - 2.5x_4}$$

eliminate from the other equations

$$1.5 + 1.5x_2 - x_3 - 2.5x_4 - x_2 + x_3 + 2x_4 = 1$$

$$\boxed{+0.5x_2 + 0x_3 - 0.5x_4 = -0.5}$$

$$3(1.5 + 1.5x_2 - x_3 - 2.5x_4)$$

$$4.5 + 4.5x_2 - 3x_3 - 7.5x_4 + 2x_2 + 2x_3 - x_4 = 0$$

$$6.5x_2 - x_3 - 8.5x_4 = -4.5$$

$$1.5 + 1.5x_2 - x_3 - 2.5x_4 + x_2 - 3x_3 - x_4 = 0$$

$$2.5x_2 - 4x_3 - 3.5x_4 = -1.5$$

$$\begin{array}{l} 0.5x_2 + 0x_3 - 0.5x_4 = -0.5 \\ 6.5x_2 - x_3 - 8.5x_4 = -4.5 \\ 2.5x_2 - 4x_3 - 3.5x_4 = -1.5 \end{array}$$

Eliminate ⇒ solve for x_2

$$\boxed{x_2 = -1 + x_4}$$

substitute in the rest

$$-6.5 + 6.5x_4 - x_3 - 8.5x_4 = -4.5$$

$$\boxed{x_3 = -2x_4 + 2}$$

$$+8x_4 + 8 - x_4 = 1$$

$$\begin{array}{l} -x_3 - 2.0x_4 = +2 \\ -4x_3 - x_4 = +1 \end{array}$$

$$-2.5 + 2.5x_4 - 4x_3 - 3.5x_4 = -1.5$$

$$\boxed{7x_4 = -7}$$

Matrix interpretation of Gaussian elimination

③ GAUSSIAN ELIMINATION: OUTLINE OF TOPICS

The rest of this section (and part of the next) will be conc with solving linear equation systems, primarily by Gaussian elimination.

Here is an outline of the topics to be covered:

a: Matrix and vector notation

b: Matrix interpretation of Gaussian elimination

c: Cost of Gaussian elimination

d: Does Gaussian elimination always work?

Answer: No.
Modifications to make it work, when possible.

e: Does Gaussian elimination work on a computer?

Answer: Only if you're careful.
Modifications to make the computations "stable".

f: Classical theory of linear equation systems

Conditions under which a solution exists
Conditions under which exactly one solution exists

MATRIX-VECTOR MULTIPLICATION

a: use shorthand $\boxed{Ax = b}$ to represent

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots \\
a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n
\end{aligned}
$$

b: A is an $n \times n$ matrix; x, b are n-vectors

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

for instance, in the preceding example,

$$
A = \begin{bmatrix} 2 & -3 & 2 & 5 \\ 1 & -1 & 1 & 2 \\ 3 & 2 & 2 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}
$$

solution vector

c: definition: if A an $n \times n$ matrix, x an n-vector,
the ith element of $Ax = (Ax)_i = \sum_{j=1}^{n} a_{ij} x_j \qquad (i = 1, \dots, n)$

d: two ways to think about it:

(i) $(Ax)_i$ is inner product of x and ith row of A

(ii) Ax is a weighted sum of the columns of A: weight on jth column is x_j

⑤ MATRIX FORM OF AN ELIMINATION STEP

a: in example, first elimination step takes

$$b = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 \\ -0.5 \\ -4.5 \\ -1.5 \end{bmatrix}$$

by subtracting 0.5, 1.5, 0.5 times b_1 from b_2, b_3, & b_4

b: what is a matrix E such that first elimination step takes $b \rightarrow Eb$?

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 \\ -1.5 & 0 & 1 & 0 \\ -0.5 & 0 & 0 & 1 \end{bmatrix}$$

MOTIVATION OF MATRIX MULTIPLICATION

a: since $\underline{Ax=b}$, it had better be $\underline{\text{true that}}$ $E(Ax) = Eb$

b: but notice also that, analogously to above, first elimination step takes

$$A = \begin{bmatrix} 2 & -3 & 2 & 5 \\ 1 & -1 & 1 & 2 \\ 3 & 2 & 2 & -1 \\ 1 & 1 & -3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -3 & 2 & 5 \\ 0 & 0.5 & 0 & -0.5 \\ 0 & 6.5 & -1 & -8.5 \\ 0 & 2.5 & -4 & -3.5 \end{bmatrix}$$

by subtracting 0.5, 1.5, 0.5 times first row from 2nd, 3rd, 4th rows

c: so, by analogy, want to say that elimination takes $b \to Eb$ and $A \to EA$, yielding $(EA)x = Eb$

d: conclusion — must define matrix product EA so that $E(Ax) = (EA)x$

e:
$$\begin{aligned} [E(Ax)]_i &= \Sigma_{j=1}^{n} e_{ij} (Ax)_j \\ &= \Sigma_{j=1}^{n} (e_{ij} \Sigma a_{jk} x_k) \\ &= \Sigma_{k=1}^{n} (\Sigma_{j=1}^{n} e_{ij} a_{jk}) x_k \end{aligned}$$

$$[(EA)x]_i = \Sigma_{k=1}^{n} (EA)_{ik} x_k$$

THUS, to have $E(Ax) = (EA)x$, it is necessary to define EA by

$$(EA)_{ik} = \Sigma_{j=1}^{n} e_{ij} a_{jk}$$

MATRIX MULTIPLICATION

a: ways to think of it —

$(AB)_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$

$(AB)_{ik}$ = inner product of ith <u>row</u> of A with kth <u>column</u> of B

kth column of AB is A times the kth column of B

b: <u>associative</u> — $A(BC) = (AB)C$

c: <u>distributive</u> — $A(B+C) = AB + AC$
$(B+C)D = BD + CD$

d: <u>not commutative</u> — in general, $AB \neq BA$

e: to come later:

similar definitions for non-square matrices
matrix analogues of 1 (<u>identity</u>) and $\frac{1}{a}$ (<u>inverse</u>)

(6A) ALTERNATE MOTIVATION

a: notice that elimination operates on each column of
A exactly as it does on b

b: so define the product EA as follows:

if the kth column of A is A_k,
then the kth column of EA is EA_k

(where A_j is treated as an n-vector)

c: $$
\begin{aligned}
(EA)_{ik} &= (EA_k)_i \\
&= \Sigma_{j=1}^{n} e_{ij} (A_k)_j \\
&= \Sigma_{j=1}^{n} e_{ij} a_{jk}
\end{aligned}
$$

SAME RESULT AS BEFORE

⑧ MATRICES AND ELIMINATION

a: original system —

$$A = \begin{bmatrix} 2 & -3 & 2 & 5 \\ 1 & -1 & 1 & 2 \\ 3 & 2 & 2 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

b: first elimination step —

subtract $\ell_{21} = 0.5$ times 1st equation from 2nd
subtract $\ell_{31} = 1.5$ times 1st equation from 3rd
subtract $\ell_{41} = 0.5$ times 1st equation from 4th

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 \\ -1.5 & 0 & 1 & 0 \\ -0.5 & 0 & 0 & 1 \end{bmatrix} \qquad E_1 A = \begin{bmatrix} 2 & -3 & 2 & 5 \\ 0 & 0.5 & 0 & -0.5 \\ 0 & 6.5 & -1 & -8.5 \\ 0 & 2.5 & -4 & -3.5 \end{bmatrix} \qquad E_1 b = \begin{bmatrix} 3 \\ -0.5 \\ -4.5 \\ -1.5 \end{bmatrix}$$

c: second elimination step —

subtract $\ell_{32} = 13$ times 2nd equation from 3rd
subtract $\ell_{42} = 5$ times 2nd equation from 4th

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -13 & 1 & 0 \\ 0 & -5 & 0 & 1 \end{bmatrix} \qquad E_2 E_1 A = \begin{bmatrix} 2 & -3 & 2 & 5 \\ 0 & 0.5 & 0 & -0.5 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -4 & -1 \end{bmatrix} \qquad E_2 E_1 b = \begin{bmatrix} 3 \\ -0.5 \\ 2 \\ 1 \end{bmatrix}$$

d: third elimination step —

subtract $\ell_{43} = 4$ times 3rd equation from 4th

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{bmatrix} \qquad E_3 E_2 E_1 A = \begin{bmatrix} 2 & -3 & 2 & 5 \\ 0 & 0.5 & 0 & -0.5 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 7 \end{bmatrix} \qquad E_3 E_2 E_1 b = \begin{bmatrix} 3 \\ -0.5 \\ 2 \\ -7 \end{bmatrix}$$

(8) e: summary — elimination took $Ax = b$ to

$$E_3 E_2 E_1 A x = E_3 E_2 E_1 b \quad \equiv \quad Ux = c$$

where U is <u>upper triangular</u> — all zero below diagonal

f: hence elimination effectively transforms an arbitrary linear system to an <u>equivalent</u> upper-triangular system

<u>and</u> upper-triangular systems are trivial to solve, by back-substitution

⑨ ELIMINATION AS TRIANGULAR FACTORIZATION

a: since $Ax = b$ and $Ux = c$ are equivalent,
can consider running the whole elimination <u>in reverse</u> —

b: undo the <u>third</u> elimination step —
<u>add</u> $\ell_{43} = 4$ times 3rd equation to 4th

$$F_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix} \qquad F_3 U x = F_3 c$$

c: undo the <u>second</u> elimination step —
add $\ell_{32} = 13$ times 2nd equation to 3rd
add $\ell_{42} = 5$ times 2nd equation to 4th

$$F_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 13 & 1 & 0 \\ 0 & 5 & 0 & 1 \end{bmatrix} \qquad F_2 F_3 U x = F_2 F_3 c$$

d: undo the <u>first</u> elimination step —
add $\ell_{21} = 0.5$ times 1st equation to 2nd
add $\ell_{31} = 1.5$ times 1st equation to 3rd
add $\ell_{41} = 0.5$ times 1st equation to 4th

$$F_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 1.5 & 0 & 1 & 0 \\ 0.5 & 0 & 0 & 1 \end{bmatrix} \qquad F_1 F_2 F_3 U x = F_1 F_2 F_3 c$$

(8)

15 Last

⑨ e: the elimination is now entirely undone, so it must be that

$$F_1 F_2 F_3 U = A$$

f: $F_1 F_2 F_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 1.5 & 13 & 1 & 0 \\ 0.5 & 5 & 4 & 1 \end{bmatrix} = L$.

L is <u>unit lower triangular</u>, and
L's below-diagonal elements are exactly the multipliers ℓ_{ij} !

g: conclusion —
Gaussian elimination finds a factorization $\boxed{A = LU}$
where L is unit lower-△, U is upper-▽

h: similar factorizations —

L is lower-△, U is unit upper-▽

B = LDU, L is unit lower-△, U is unit upper-▽,
D is <u>diagonal</u>

$$\begin{bmatrix} \cdots \\ 1 & 1 & -3 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

```
>> A = - - -
>> [L U] = lu(sym(A))
```

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 3/2 & 13 & 1 & 0 \\ 1/2 & 5 & 4 & 1 \end{bmatrix} \qquad L \begin{bmatrix} 1 & & 0 \\ X & 1 & \\ & X & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & -3 & 2 & 5 \\ 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 7 \end{bmatrix} \qquad U = \begin{bmatrix} & X \\ 0 & \end{bmatrix}$$

```
>> X = vpa(A\b, 4)
```

$$X = \begin{bmatrix} 1 \\ -2 \\ 0 \\ -1 \end{bmatrix}$$