

Gaussian Elimination

Major: All Engineering Majors

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Transforming Numerical Methods Education for STEM
Undergraduates

Naïve Gauss Elimination

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Naïve Gaussian Elimination

A method to solve simultaneous linear equations of the form $[A][X]=[C]$

Two steps

1. Forward Elimination
2. Back Substitution

Forward Elimination

The goal of forward elimination is to transform the coefficient matrix into an upper triangular matrix



Forward Elimination

A set of n equations and n unknowns

$\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix}$

$(n - 1)$ steps of forward elimination

Forward Elimination

Step 1

For Equation 2, divide Equation 1 by and multiply by .

Forward Elimination

Subtract the result from Equation 2.

—

or

Forward Elimination

Repeat this procedure for the remaining equations to reduce the set of equations as

$$\begin{array}{cc} \cdot & \cdot \cdot \\ \cdot & \cdot \cdot \\ \cdot & \cdot \cdot \end{array}$$

End of Step 1

Forward Elimination

Step 2

Repeat the same procedure for the 3rd term of Equation 3.

⋮ ⋮

End of Step 2

Forward Elimination

At the end of $(n-1)$ Forward Elimination steps, the system of equations will look like

$$\begin{array}{cccc} \cdot & \cdot & & \\ & & \cdot & \cdot \\ & & & \cdot \end{array}$$

End of Step $(n-1)$

Matrix Form at End of Forward Elimination

Back Substitution

Solve each equation starting from the last equation

Example of a system of 3 equations

Back Substitution Starting Eqns

.

.

.

.

Back Substitution

Start with the last equation because it has only one unknown

Back Substitution

THE END

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Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/gaussian_elimination.html

Naïve Gauss Elimination Example

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Example 1

The upward velocity of a rocket is given at three different times

Table 1 Velocity vs. time data.

Time,	Velocity,
5	106.8
8	177.2
12	279.2



The velocity data is approximated by a polynomial as:

Find the velocity at $t=6$ seconds .

Example 1 Cont.

Assume

Results in a matrix template of the form:

Using data from Table 1, the matrix becomes:

Example 1 Cont.

- Forward Elimination
- Back Substitution

Forward Elimination

Number of Steps of Forward Elimination

Number of steps of forward elimination is
 $(n - 1) = (3 - 1) = 2$

Forward Elimination: Step 1

Divide Equation 1 by 25 and
multiply it by 64, .

Subtract the result from
Equation 2

Substitute new equation for
Equation 2

Forward Elimination: Step 1 (cont.)

Divide Equation 1 by 25 and
multiply it by 144, .

Subtract the result from
Equation 3

Substitute new equation for
Equation 3

Forward Elimination: Step 2

Divide Equation 2 by -4.8
and multiply it by
 -16.8 ,

.

Subtract the result from
Equation 3

Substitute new equation for
Equation 3

Back Substitution

Back Substitution

Solving for a_3

Back Substitution (cont.)

Solving for a_2

Back Substitution (cont.)

Solving for a_1

Naïve Gaussian Elimination Solution

Example 1 Cont.

Solution

The solution vector is

The polynomial that passes through the three data points is then:

THE END

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Naïve Gauss Elimination Pitfalls

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Pitfall#1. Division by zero

Is division by zero an issue here?

Is division by zero an issue here?
YES



Division by zero is a possibility at any step
of forward elimination

Pitfall#2. Large Round- off Errors

Exact Solution

Pitfall#2. Large Round- off Errors

Solve it on a computer using **6** significant digits with chopping

Pitfall#2. Large Round- off Errors

Solve it on a computer using **5** significant digits with chopping

Is there a way to reduce the round off error?

Avoiding Pitfalls

Increase the number of significant digits

- Decreases round- off error
- Does not avoid division by zero

Avoiding Pitfalls

Gaussian Elimination with Partial Pivoting

- Avoids division by zero
- Reduces round off error

THE END

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Gauss Elimination with Partial Pivoting

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Pitfalls of Naïve Gauss Elimination

- Possible division by zero
- Large round-off errors

Avoiding Pitfalls

Increase the number of significant digits

- Decreases round- off error
- Does not avoid division by zero

Avoiding Pitfalls

Gaussian Elimination with Partial Pivoting

- Avoids division by zero
- Reduces round off error

What is Different About Partial Pivoting?

At the beginning of the k^{th} step of forward elimination, find the maximum of

If the maximum of the values
is in the p^{th} row, then switch rows p and k .

Matrix Form at Beginning of 2nd Step of Forward Elimination

Example (2nd step of FE)

Which two rows would you switch?

Example (2nd step of FE)

Switched Rows

Gaussian Elimination with Partial Pivoting

A method to solve simultaneous linear equations of the form $[A][X]=[C]$

Two steps

1. Forward Elimination
2. Back Substitution

Forward Elimination

Same as naïve Gauss elimination method except that we switch rows before each of the $(n-1)$ steps of forward elimination.

Example: Matrix Form at Beginning of 2nd Step of Forward Elimination

Matrix Form at End of Forward Elimination

Back Substitution Starting Eqns

.

.

.

.

Back Substitution

THE END

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Gauss Elimination with Partial Pivoting Example

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Example 2

Solve the following set of equations by Gaussian elimination with partial pivoting

Example 2 Cont.

- Forward Elimination
- Back Substitution

Forward Elimination

Number of Steps of Forward Elimination

Number of steps of forward elimination is
 $(n-1)=(3-1)=2$

Forward Elimination: Step 1

- Examine absolute values of first column, first row and below.
- Largest absolute value is 144 and exists in row 3.
- Switch row 1 and row 3.

Forward Elimination: Step 1 (cont.)

Divide Equation 1 by 144 and
multiply it by 64, .

Subtract the result from
Equation 2

Substitute new equation for
Equation 2

Forward Elimination: Step 1 (cont.)

Divide Equation 1 by 144 and
multiply it by 25, .

Subtract the result from
Equation 3

Substitute new equation for
Equation 3

Forward Elimination: Step 2

- Examine absolute values of second column, second row and below.
- Largest absolute value is 2.917 and exists in row 3.
- Switch row 2 and row 3.

Forward Elimination: Step 2 (cont.)

Divide Equation 2 by 2.917 and multiply it by 2.667,

Subtract the result from
Equation 3

Substitute new equation for
Equation 3

Back Substitution

Back Substitution

Solving for a_3

Back Substitution (cont.)

Solving for a_2

Back Substitution (cont.)

Solving for a_1

Gaussian Elimination with Partial Pivoting Solution

Gauss Elimination with Partial Pivoting Another Example

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Partial Pivoting: Example

Consider the system of equations

In matrix form

=

Solve using Gaussian Elimination with Partial Pivoting using five significant digits with chopping

Partial Pivoting: Example

Forward Elimination: Step 1

Examining the values of the first column

$|10|$, $|-3|$, and $|5|$ or 10, 3, and 5

The largest absolute value is 10, which means, to follow the rules of Partial Pivoting, we switch row1 with row1.

Performing Forward Elimination

Partial Pivoting: Example

Forward Elimination: Step 2

Examining the values of the first column

$|-0.001|$ and $|2.5|$ or 0.0001 and 2.5

The largest absolute value is 2.5 , so row 2 is switched with row 3

Performing the row swap

Partial Pivoting: Example

Forward Elimination: Step 2

Performing the Forward Elimination results in:

Partial Pivoting: Example

Back Substitution

Solving the equations through back substitution

Partial Pivoting: Example

Compare the calculated and exact solution

The fact that they are equal is coincidence, but it does illustrate the advantage of Partial Pivoting

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Determinant of a Square Matrix Using Naïve Gauss Elimination

Example

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Theorem of Determinants

If a multiple of one row of $[A]_{n \times n}$ is added or subtracted to another row of $[A]_{n \times n}$ to result in $[B]_{n \times n}$ then $\det(A) = \det(B)$

Theorem of Determinants

The determinant of an upper triangular matrix $[A]_{n \times n}$ is given by

Forward Elimination of a Square Matrix

Using forward elimination to transform $[A]_{n \times n}$ to an upper triangular matrix, $[U]_{n \times n}$.

Example

Using naïve Gaussian elimination find the determinant of the following square matrix.

Forward Elimination

Forward Elimination: Step 1

Divide Equation 1 by 25 and
multiply it by 64, .

Subtract the result from
Equation 2

Substitute new equation for
Equation 2

Forward Elimination: Step 1 (cont.)

Divide Equation 1 by 25 and
multiply it by 144, .

Subtract the result from
Equation 3

Substitute new equation for
Equation 3

Forward Elimination: Step 2

Divide Equation 2 by -4.8
and multiply it by
 -16.8 ,

.

Subtract the result from
Equation 3

Substitute new equation for
Equation 3

Finding the Determinant

After forward elimination

.

Summary

- Forward Elimination
- Back Substitution
- Pitfalls
- Improvements
- Partial Pivoting
- Determinant of a Matrix

Additional Resources

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