LU Decomposition

Major: All Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

LU Decomposition

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LU Decomposition

LU Decomposition is another method to solve a set of simultaneous linear equations

Which is better, Gauss Elimination or LU Decomposition?

To answer this, a closer look at LU decomposition is needed.

LU Decomposition

Method

For most non- singular matrix [A] that one could conduct Naïve Gauss Elimination forward elimination steps, one can always write it as

$$[A] = [L][U]$$

where

[L] = lower triangular matrix

[U] = upper triangular matrix

How does LU Decomposition work?

If solving a set of linear equations	$[A][X] = [C]$
If $[A] = [L][U]$ then	$[L][U][X] = [C]$
Multiply by	$[L]^{-1}$
Which gives	$[L]^{-1}[L][U][X] = [L]^{-1}[C]$
Remember $[L]^{-1}[L] = [I]$ which leads to	$[I][U][X] = [L]^{-1}[C]$
Now, if $[I][U] = [U]$ then	$[U][X] = [L]^{-1}[C]$
Now, let	$[L]^{-1}[C]=[Z]$
Which ends with	$[L][Z] = [C] \quad (1)$
and	$[U][X] = [Z] \quad (2)$

LU Decomposition

How can this be used?

Given [A][X] = [C]

- Decompose [A] into [L] and [U]
- Solve [L][Z] = [C] for [Z]
- Solve [U][X] = [Z] for [X]

Is LU Decomposition better than Gaussian Elimination?

Solve [A][X] = [B]

T = clock cycle time and nxn = size of the matrix

Forward Elimination

Back Substitution

Decomposition to LU

Forward Substitution

Back Substitution

Is LU Decomposition better than Gaussian Elimination?

To solve [A][X] = [B]

Time taken by methods

Gaussian Elimination	LU Decomposition

T = clock cycle time and nxn = size of the matrix

So both methods are equally efficient.

To find inverse of [A]

Time taken by Gaussian Elimination Time taken by LU Decomposition

To find inverse of [A]

Time taken by Gaussian Elimination Time taken by LU Decomposition

Table 1 Comparing computational times of finding inverse of a matrix using LU decomposition and Gaussian elimination.

n	10	100	1000	10000
$CT\|_{inverse\ GE} / CT\|_{inverse\ LU}$	3.288	25.84	250.8	2501

For large n, $CT|_{inverse\ GE} / CT|_{inverse\ LU} \approx n/4$

Method: [A] Decomposes to [L] and [U]

[U] is the same as the coefficient matrix at the end of the forward elimination step.

[L] is obtained using the multipliers that were used in the forward elimination process

Finding the [U] matrix

Using the Forward Elimination Procedure of Gauss Elimination

Step 1:

Finding the [U] Matrix

Matrix after Step 1:

Step 2:

Finding the [L] matrix

Using the multipliers used during the Forward Elimination Procedure

From the first step
of forward
elimination

Finding the [L] Matrix

From the second step of forward elimination

Does [L][U] = [A]?

?

Using LU Decomposition to solve SLEs

Solve the following set of linear equations using LU Decomposition

Using the procedure for finding the [L] and [U] matrices

Example

Set [L][Z] = [C]

Solve for [Z]

Example

Complete the forward substitution to solve for [Z]

Example

Set [U][X] = [Z]

Solve for [X]   The 3 equations become

Example

From the 3rd equation

Substituting in a_3 and using the second equation

Example

Substituting in a_3 and a_2 using the first equation

Hence the Solution Vector is:

Finding the inverse of a square matrix

The inverse [B] of a square matrix [A] is defined as

$$[A][B] = [I] = [B][A]$$

Finding the inverse of a square matrix

How can LU Decomposition be used to find the inverse?

Assume the first column of [B] to be $[b_{11} \;\; b_{12} \;\; \ldots \;\; b_{n1}]^T$

Using this and the definition of matrix multiplication

First column of [B] Second column of [B]

The remaining columns in [B] can be found in the same manner

Example: Inverse of a Matrix

Find the inverse of a square matrix [A]

Using the decomposition procedure, the [L] and [U] matrices are found to be

Example: Inverse of a Matrix

Solving for the each column of [B] requires two steps

- Solve [L] [Z] = [C] for [Z]
- Solve [U] [X] = [Z] for [X]

Step 1:

This generates the equations:

Example: Inverse of a Matrix

Solving for [Z]

Example: Inverse of a Matrix

Solving [U][X] = [Z] for [X]

Example: Inverse of a Matrix

Using Backward Substitution

So the first column of the inverse of [A] is:

Example: Inverse of a Matrix

Repeating for the second and third columns of the inverse

Second Column   Third Column

Example: Inverse of a Matrix

The inverse of [A] is

To check your work do the following operation

$$[A][A]^{-1} = [I] = [A]^{-1}[A]$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple- choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/lu_decomposition.html

THE END

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