Based on previous equations we can do the following:

Maximum Velocity:

In the expression for the dependence of velocity on position, we can see that the maximum velocity occurs at

$r = 0$ ie at the center of the pipe

$$\therefore \boxed{V_{x,max} = \frac{\Delta P r_0^2}{4 \mu \Delta x}} \Rightarrow V_x = V_{x,max}\left[1 - \left(\frac{r}{r_0}\right)^2\right]$$

Average Velocity:

By definition

$$V_{ave} = \frac{\int V_x \, dA}{\int dA} \quad ; \text{ remember } dA = r\,dr\,d\theta.$$

$$= \frac{V_{x,max} \int_0^{2\pi}\int_0^{r_0} \left[1 - \left(\frac{r}{r_0}\right)^2\right] r\,dr\,d\theta}{\int_0^{2\pi}\int_0^{r_0} r\,dr\,d\theta}$$

$$= \frac{V_{x,max} \left[\frac{r^2}{2} - \frac{r^4}{4r_0^2}\right]\Big|_0^{r_0}}{\frac{r^2}{2}\Big|_0^{r_0}}$$

$$= \frac{V_{x,max}\left[\frac{r_0^2}{2} - \frac{r_0^4}{4r_0^2}\right]}{\frac{r_0^2}{2}} = \frac{V_{x,max}\left[\frac{r_0^2}{2} - \frac{r_0^2}{4}\right]}{\frac{r_0^2}{2}}$$

$$= \frac{V_{x,max}\left[\frac{1}{2} - \frac{1}{4}\right]}{\frac{1}{2}}$$

$$\boxed{V_{ave} = \frac{V_{x,max}}{2}}$$

Volumetric flow rate, Q

$$Q = \int v_x \, dA$$

$$= \int_0^{2\pi}\int_0^{r_0} v_x \, r\,dr\,d\theta \quad = \int_0^{2\pi}\int_0^{r_0} \frac{\Delta P r_0^2}{4\mu\Delta x}\left[1-\left(\frac{r}{r_0}\right)^2\right] r\,dr\,d\theta$$

$$= \frac{\Delta P r_0^2}{4\mu\Delta x} \, 2\pi \, \frac{r_0^2}{4_2}$$

$$\boxed{\therefore \quad Q = \frac{\pi \Delta P r_0^4}{8\mu\,\Delta x} = \frac{\pi \Delta P D^4}{128\,\mu\,\Delta x}}$$

Hagen - Poiseuille Equation.

$\therefore$ for laminar flow we can see that

$$\boxed{\Delta P \propto Q}$$

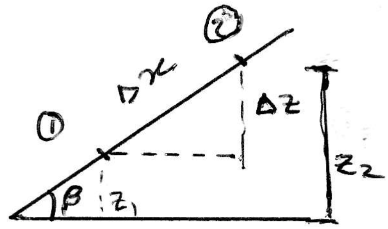
We can see that

$$V_{ave} = \frac{Q}{\pi r_0^2} = \frac{V_{max}}{2}$$

Friction heating $\mathcal{F}$:

From BE

$$\frac{P_2 - P_1}{\rho} + g(z_2 - z_1) = -\mathcal{F}$$



$$\sin\beta = \frac{\Delta z}{\Delta x}$$

$$\Delta z = \Delta x \sin\beta$$

$$-\left(\frac{P_1 - P_2}{\rho}\right) + g(z_2 - z_1) = -\mathcal{F}$$

$$\therefore \; -\left[\frac{(P_1 - P_2)}{\rho} - g\,\Delta x \sin\beta\right] = -\mathcal{F}$$

$$-\left[\frac{(P_1 - P_2) - \rho g \Delta x \sin\beta}{\rho}\right] = -\mathcal{F}$$

$$-\frac{\Delta \mathcal{P}}{\rho} = -\mathcal{F}$$

$$\Rightarrow \quad \boxed{\frac{\Delta \mathcal{P}}{\rho} = \mathcal{F}}$$

It follows that from Hagen equation, we can show that

$$\boxed{\mathcal{F} = Q \,\Delta x\, \frac{\mu}{\rho}\, \frac{128}{\pi D^4}}$$

This equation applies to a pipe with any inclination

Think about: dependence of $\mathcal{F}$ on Q, length (Δx) and diameter?

Note:

From force balance (mentioned previously) we found

$$\tau = \frac{1}{2}\, \frac{\Delta P}{\Delta x} \cdot r$$

$$\Rightarrow \quad \Delta P = 2\tau \Delta x \cdot \frac{1}{r}$$

$$\text{BE} \Rightarrow \quad \frac{\Delta P}{\rho} = \mathcal{F} \quad \Rightarrow \quad \Delta P = \rho \mathcal{F}$$

$$\Rightarrow \quad \boxed{\mathcal{F} = \frac{2\tau \Delta x}{\rho} \cdot \frac{1}{r}}$$

This equation applies to any flow.

It can be seen that the maximum friction occurs when $r = r_0$ ie at the walls of the pipe.

Turbulent flow:

From force balance $\Delta P = \dfrac{2\tau \Delta x}{r}$ which is valid for any type of flow.

and $\mathcal{F} = \dfrac{2\tau \Delta x}{\rho} \cdot \dfrac{1}{r}$

Problem is how to account for τ

$\tau_t > \tau_L$ due to movement in more than one direction (specially $\perp$ axe of bulk motion)

The difference between τ_t and τ_L is called "Reynolds Stress"

$$\therefore \quad \mathcal{F}_{turbulent} > \mathcal{F}_{Laminar}$$

In turbulent flow, the shear stress at the wall τ_w occurs when $r = r_o$

$$\therefore \quad \Delta P = P_1 - P_2 = \frac{2\Delta x}{r_o}\tau_w = \rho \mathcal{F}$$

$$= \frac{4\Delta x}{D} \cdot \tau_w$$

for turbulent flow $\tau_w \propto \rho \frac{V^2}{2}$ (V: average velocity)

$$\therefore \tau_w = f\left(\tfrac{1}{2}\rho V^2\right) \quad ; \quad \tfrac{1}{2}\rho V^2 = \tfrac{1}{2}\frac{mV^2}{Q}$$

$= \dfrac{\text{Kinetic energy}}{\text{vol. flow rate.}}$

substitute:

$$\boxed{\Delta P = \frac{\Delta x f\left(\frac{1}{2}\rho V^2\right)}{D}}$$

DARCY (UK)
or
FANNING (USA)
Equation.

Since $\Delta P = \rho \mathcal{F}$

$$\therefore \boxed{f = \frac{\mathcal{F}}{4\left(\frac{\Delta x}{D}\right)\left(\frac{V^2}{2}\right)}}$$

Chemical and Mechanical Engineering definition.

f: Fanning Friction factor.

Note: Definition of friction factor.
Depending on how τ_w is related to ρV^2 we have:

Stanton and Pannel

$$\phi = \frac{\tau_w}{\rho v^2}$$

[Coulson and Richardson]

Fanning or Darcy

$$f = \frac{\tau}{\rho v^2/2} = 2\phi$$

Moody

$$f' = \frac{4\tau_w}{\rho v^2/2} = 8\phi$$

Relationship

$$\phi = \frac{f}{2} = \frac{f'}{8}$$

chem and Mech — Civil Engineering

Care must be taken !!!

f depends on
- Reynolds Number $\quad f \downarrow$ as $R \uparrow$
- Roughness of pipe "ϵ" or "$\frac{\epsilon}{D}$" = relative roughness

  $f \uparrow$ as $\frac{\epsilon}{D} \uparrow$

Friction factor for Laminar Flow:

Although not needed; however in order to unify the treatment of flow problems, specially that in some cases we do not know whether the flow is laminar or turbulent, we can define an expression for friction factor as follows:

From Hagen Equation $\Rightarrow \Delta P = \dfrac{Q \, \Delta x \, \mu \, 128}{\pi D^4}$

From Darcy Equation $\Rightarrow \Delta P = \dfrac{4 \Delta x f (\frac{1}{2} \rho V^2)}{D}$

$$\Rightarrow \quad \frac{4 \Delta x f (\frac{1}{2} \rho V^2)}{D} = \frac{Q \, \Delta x \, \mu \, 128}{\pi D^4}$$

$$\Rightarrow \quad f = \frac{64 \, Q \, \mu}{\pi \rho V^2 D^3} \quad ; \quad Q = \frac{\pi D^2}{4} \cdot V$$

$$= \frac{64 \frac{\pi D^2}{4} V \mu}{\pi \rho V^2 D^3}$$

$$= \frac{16 \, \mu}{\rho V D} \quad ; \quad Re = \frac{\rho V D}{\mu}$$

$$\boxed{f = \frac{16}{Re}}$$

and

$$\boxed{\log f = \log 16 - \log Re}$$