Topic VII: Dimensional Analysis

Dimensional analysis: definitions; Buckingham π-theorem**.**

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- **Dimensional Analysis:** re-description of an engineering problem in terms of dimensionless groups(variables) instead of dimensional ones.

- **Fundamental dimensions:**
  - Mass; M
  - Length; L
  - Time; T
  - Temperature; θ

- Remember that any equation must be dimensionally consistent.;example MEB:

$$g(z_2 - z_1) + \frac{u_2^2 - u_1^2}{2} + \frac{P_2 - P_1}{\rho} = w_p - w_f$$

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- **Benefits of dimensional analysis:**
  1. It is easier to deal with dimensionless quantities rather than dimensional ones.
  2. Number of variables sufficient to describe the problem will be reduced.

- **How to perform dimensional analysis?**

→ The most common method is **Buckingham π theorem:**

- **Buckingham π theorem**: suppose that an engineering problem is described sufficiently in terms of **n** dimensional variables (A_1, A_2,…, A_n) that have **m** fundamental dimensions. The problem can be re-described sufficiently in terms of **n-m** dimensionless variables or groups (π_1, π_2, …, π_{n-m}).

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- **How to determine the dimensionless groups π's?**

A. Select **m repeating variables** from among the original variables. These repeating variables must satisfy the following two rules:

Rule 1. All **m fundamental dimensions** must appear in the **m** repeating variables.

Rule 2. It must **NOT** be possible to form a dimensionless groups solely from these repeating variables.

B. The m repeating variables are used with each one of the other variables to specify each dimensionless group π.

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- For example let a problem has 7 dimensional variables (A_1,A_2,…, A_7) with 3 fundamental dimensions (MLT):

→ # of dimensional variables; n=7
  # of fundamental dimensions ; m=3
  # of dimensionless groups= n-m=7-3=4: π_1, π_2, π_3, π_4
  # of repeating variables=m=3

If the selected repeating variables that satisfy the two rules are (A_2,A_3,A_5). The four dimensional variables will be written as:

$$\pi_1 = A_2^{x_1} A_3^{y_1} A_5^{z_1} A_1$$

$$\pi_2 = A_2^{x_2} A_3^{y_2} A_5^{z_2} A_4$$

$$\pi_3 = A_2^{x_3} A_3^{y_3} A_5^{z_3} A_6$$

$$\pi_4 = A_2^{x_4} A_3^{y_4} A_5^{z_4} A_7$$

The exponents (x_1,…z_4) in the above equations are to be determined so that each π is dimensionless ($M^0L^0T^0$).

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Example. The relationship between variables affecting the pressure drop for flow of Newtonian fluid in a smooth pipe can be rewritten as:

$-\Delta P$=Function(D,L,ρ, μ, U)

Use Buckingham π theorem to find the dimensionless groups that describe this problem.

# of dimensional variables; n=6 ($-\Delta P$, D,L,ρ, μ, U)
# of fundamental dimensions ; m=3 (MLT)
# of dimensionless groups= n-m=6-3=3: π_1, π_2, π_3
# of repeating variables; m=3
Let us know select 3 repeating variables:

- If D,L,ρ are selected as repeating variables are.
  Rule 1 is not satisfied (T dimension does not appear in these variables. Also, **Rule 2** is not satisfied L/D or D/L is dimensionless.
  ⇒ D,L,ρ are bad choice to be repeating variables.

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- If μ, L, D are selected as repeating variables are.
  Rule 1 is satisfied (MLT dimensions appear in these variables. **Rule 2** is not satisfied L/D or D/L is dimensionless.
  ⇒ μ, L, D are bad choice to be repeating variables.

- If $-\Delta P$, U, ρ are selected as repeating variables are.
  Rule 1 is satisfied MLT dimensions appear in these variables. **Rule 2** is not satisfied (From MEB we learn that $-\Delta P/\rho$ has the same unit as U^2. Thus, $-\Delta P/(\rho U^2)$ or $\rho U^2/(-\Delta P)$ is dimensionless
  ⇒ $-\Delta P$, U, ρ are bad choice to be repeating variables.

- If μ, U, ρ are selected as repeating variables are.
  Rule 1 is satisfied(MLT dimensions appear in these variables). **Rule 2** is satisfied
  ⇒ μ, U, ρ are good choice to be repeating variables.

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- If D, U, ρ are selected as repeating variables are.
  Rule 1 is satisfied(MLT dimensions appear in these variables). **Rule 2** is satisfied
  ⇒ D, U, ρ are good choice to be repeating variables.

- If L, U, ρ are selected as repeating variables are.
  Rule 1 is satisfied(MLT dimensions appear in these variables). **Rule 2** is satisfied
  ⇒ L, U, ρ are good choice to be repeating variables.

- If $-\Delta P$, D, ρ are selected as repeating variables are.
  Rule 1 is satisfied(MLT dimensions appear in these variables). **Rule 2** is satisfied
  ⇒ $-\Delta P$, L, ρ are good choice to be repeating variables.

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- Also there are other good choices for the repeating variables
- Let us use D, U, ρ as repeating variables. The dimensionless group will be written as:

$$\pi_1 = D^{x_1} U^{y_1} \rho^{z_1} \mu$$

$$\pi_2 = D^{x_2} U^{y_2} \rho^{z_2} (-\Delta P)$$

$$\pi_3 = D^{x_3} U^{y_3} \rho^{z_3} L$$

- Now let us start findings the exponents…..

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$$\pi_1 = D^{x_1} U^{y_1} \rho^{z_1} \mu$$

$$M^0 L^0 T^0 = L^{x_1} \left(\frac{L}{T}\right)^{y_1} \left(\frac{M}{L^3}\right)^{z_1} \frac{M}{LT} = M^{(z_1+1)} L^{(x_1+y_1-3z_1-1)} T^{(-y_1-1)}$$

$$\Rightarrow z_1 + 1 = 0 \Rightarrow z_1 = -1$$

$$-y_1 - 1 = 0 \Rightarrow y_1 = -1$$

$$x_1 + y_1 - 3z_1 - 1 = 0$$

$$x_1 - 1 - 3(-1) - 1 = 0 \Rightarrow x_1 = -1$$

$$\Rightarrow \pi_1 = D^{-1} U^{-1} \rho^{-1} \mu = \frac{\mu}{DU\rho} = \frac{1}{Re}$$

$$\therefore \pi_1 = \frac{1}{Re}$$

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$$\pi_2 = D^{x_2} U^{y_2} \rho^{z_2} (-\Delta P)$$

$$M^0 L^0 T^0 = L^{x_2} \left(\frac{L}{T}\right)^{y_2} \left(\frac{M}{L^3}\right)^{z_2} \frac{M}{LT^2} = M^{(z_2+1)} L^{(x_2+y_2-3z_2-1)} T^{(-y_2-2)}$$

$$\Rightarrow z_2 + 1 = 0 \Rightarrow z_2 = -1$$

$$-y_2 - 2 = 0 \Rightarrow y_2 = -2$$

$$x_2 + y_2 - 3z_2 - 1 = 0$$

$$x_2 - 2 - 3(-1) - 1 = 0 \Rightarrow x_2 = 0$$

$$\Rightarrow \pi_2 = D^0 U^{-2} \rho^{-1} (-\Delta P) = \frac{-\Delta P}{\rho U^2}$$

$$\therefore \pi_2 = \frac{-\Delta P}{\rho U^2}$$

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$$\pi_3 = D^{x_3} U^{y_3} \rho^{z_3} L$$

$$M^0 L^0 T^0 = L^{x_3} \left(\frac{L}{T}\right)^{y_3} \left(\frac{M}{L^3}\right)^{z_3} L = M^{z_3} L^{(x_3+y_3-3z_3+1)} T^{-y_3}$$

$$\Rightarrow z_3 = 0$$

$$-y_3 = 0 \Rightarrow y_3 = 0$$

$$x_3 + y_3 - 3z_3 + 1 = 0$$

$$x_3 + 0 - 3(0) + 1 = 0 \Rightarrow x_3 = -1$$

$$\Rightarrow \pi_3 = D^{-1} U^0 \rho^0 L = \frac{L}{D}$$

$$\therefore \pi_3 = \frac{L}{D}$$

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Thus, the relationship between the dimensionless variables for flow of Newtonian fluid in smooth pipe can be rewritten as:

$$\pi_3 = \text{Function}(\pi_1 , \pi_2)$$

Where the dimensionless groups as given by:

$$\pi_1 = Re \; ; \pi_2 = \frac{-\Delta P}{\rho U^2} \; ; \pi_3 = \frac{L}{D}$$

→ Note that the variables is reduced from 6 ($-\Delta P$, D,L,ρ, μ, U) to 3 (π_1, π_2, π_3).

Exercise. repeat the previous example with rough pipe.

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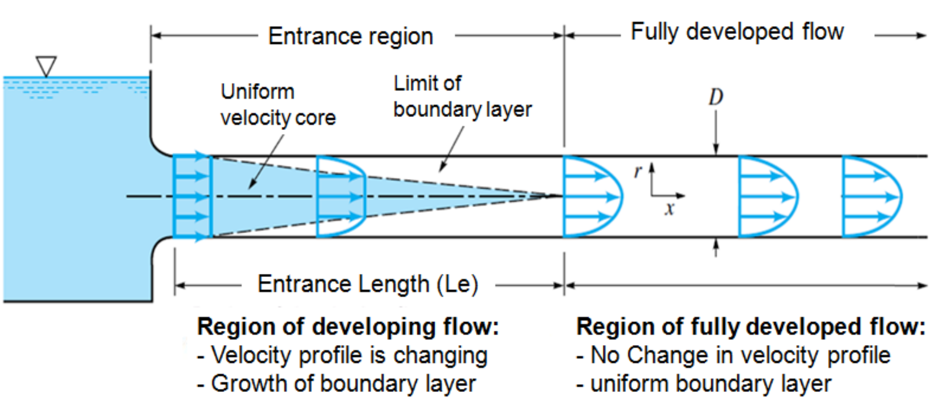
Fully developed flow in pipe:



- In such flow situations, friction factor and heat/mass transfer coefficients in the developing flow region can NOT be estimated using the same correlations of the developed flow region.

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Fully developed flow in pipe:

Example. The entrance length, L_e, for flow of Newtonian fluid in smooth pipe to be developed is function of density, average velocity, pipe diameter, and fluid viscosity:

Function(L_e, ρ, U, D, μ) = 0

Find the dimensionless parameters that describe this entry length problem.

# of dimensional variables; n = 5 (L_e, D, ρ, μ, U)
# of fundamental dimensions ; m=3 (MLT)
# of dimensionless groups= n-m=5-3=2: π_1, π_2
# of repeating variables; m=3

D, U, ρ are good choice to be repeating variables.

$$\pi_1 = D^{x_1} U^{y_1} \rho^{z_1} \mu$$

$$\pi_2 = D^{x_2} U^{y_2} \rho^{z_2} L_e$$

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Fully developed flow in pipe:

$$\pi_1 = D^{x_1} U^{y_1} \rho^{z_1} \mu$$

$$M^0 L^0 T^0 = L^{x_1} \left(\frac{L}{T}\right)^{y_1} \left(\frac{M}{L^3}\right)^{z_1} \frac{M}{LT} = M^{(z_1+1)} L^{(x_1+y_1-3z_1-1)} T^{(-y_1-1)}$$

$$\Rightarrow z_1 + 1 = 0 \Rightarrow z_1 = -1$$

$$-y_1 - 1 = 0 \Rightarrow y_1 = -1$$

$$x_1 + y_1 - 3z_1 - 1 = 0$$

$$x_1 - 1 - 3(-1) - 1 = 0 \Rightarrow x_1 = -1$$

$$\therefore \pi_1 = D^{-1} U^{-1} \rho^{-1} \mu = \frac{\mu}{DU\rho} = \frac{1}{Re}$$

$$\therefore \pi_1 = \frac{1}{Re}$$

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Fully developed flow in pipe:

$$\pi_2 = D^{x_2} U^{y_2} \rho^{z_2} L_e$$

$$M^0 L^0 T^0 = L^{x_2} \left(\frac{L}{T}\right)^{y_2} \left(\frac{M}{L^3}\right)^{z_2} L = M^{z_2} L^{(x_2 + y_2 - 3z_2 + 1)} T^{-y_2}$$

$$\Rightarrow z_2 = 0$$

$$-y_2 = 0 \Rightarrow y_2 = 0$$

$$x_2 + y_2 - 3z_2 + 1 = 0$$

$$x_2 + 0 - 3(0) + 1 = 0 \Rightarrow x_2 = -1$$

$$\therefore \pi_2 = D^{-1} U^0 \rho^0 L_e = \frac{L_e}{D}$$

$$\therefore \pi_2 = \frac{L_e}{D} \qquad \Rightarrow \pi_2 = \frac{L_e}{D} = \text{Function}(\pi_1 = Re)$$

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Fully developed flow in pipe:

- For laminar flow in pipe , the entrance length, L_e, can be estimated from the equation: **$L_e/D = 0.06\ Re$**
- For turbulent flow in pipe, the entrance length, L_e, can be estimated from the equation: **$L_e/D = 4.4Re^{1/6}$**
- For flow in conduits, use the concept of hydraulic radius or the equivalent diameter to estimate the entrance length.
- Note that the entrance length of turbulent flow will be shorter than that of laminar flow.