University of Jordan
Chemical Engineering Department
905509 Statistical Quality Control

CUSUM & EWMA Charts

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Outline

- Pitfalls of Shewhart charts
- Alternatives to Shewhart charts
  - CUSUM
  - EWMA
- Designing CUSUM and EWMA charts

Pitfalls of Shewhart Charts

- The major disadvantage of the Shewhart charts is that it only uses the information about the process contained in the last plotted point which is fine for the majority of applications.
- Two effective alternatives to the Shewhart control charts are:
  - the cumulative sum (CUSUM) control chart, and
  - the exponentially weighted moving average (EWMA) control chart.
- These two alternatives are especially useful when **small shifts** are desired to be detected.

CUSUM Control Charts

- The CUSUM chart incorporates all information in the sequence of sample values by plotting the cumulative sums of the deviations of the sample values from a target value.
- If μ_0 is the target for the process mean, $\bar{x}_j$ is the average of the j^{th} sample, then the cumulative sum control chart is formed by plotting the quantity

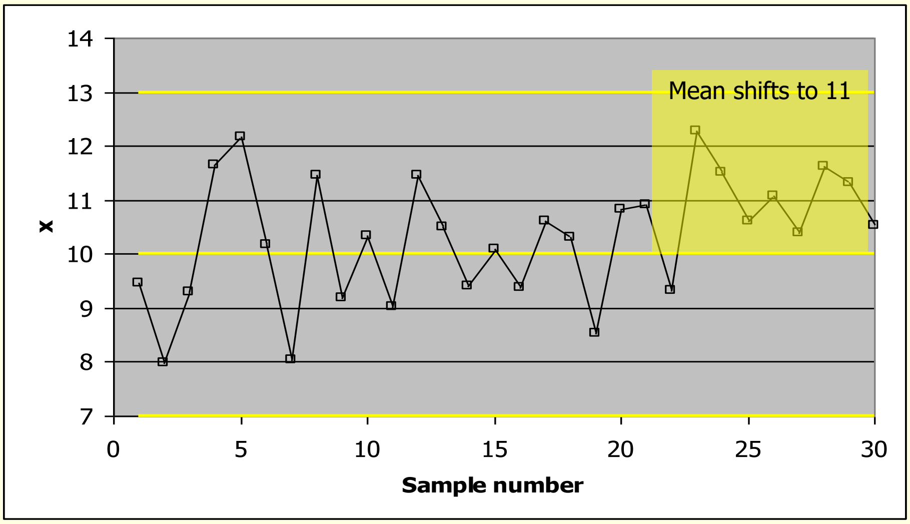
$$C_i = \sum_{j=1}^{i} (\bar{x}_j - \mu_0)$$

- If process remains in control at target value, μ_0 then CUSUM is a "random walk" with
  - Mean = zero
  - If mean shifts upward, will see an upward drift.
  - If mean shifts downward, will see a downward shift.

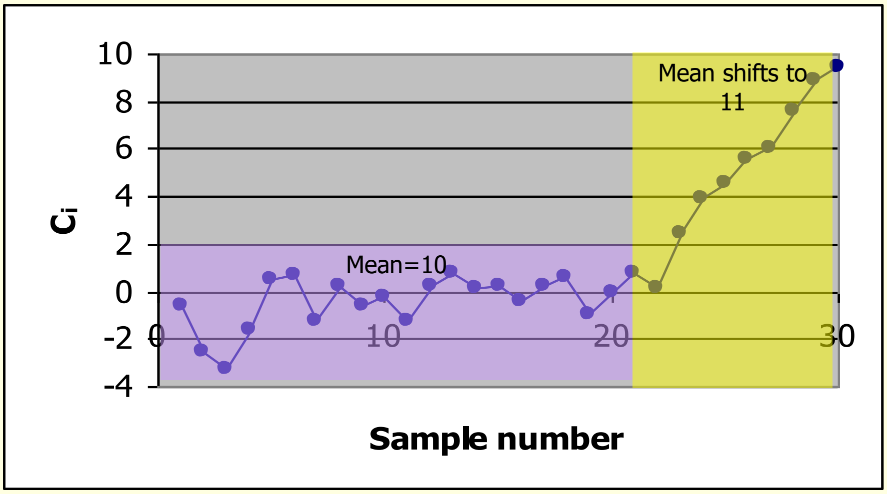
Example

- The provided Excel sheet has data taken from a normal distribution with a mean of 10 and standard deviation of 1. Construct a cusum chart after checking that a drift is occurring in the process.

Shewhart Plot



CUSUM Plot



Tabular (Algorithmic) CUSUM

- Let x_i be the i^{th} observation on the process
- If the process is in control then
  - Assume σ is known or can be estimated.
  - Accumulate deviations from the target μ_0 above the target with one statistic, C^+
  - Accumulate deviations from the target μ_0 below the target with another statistic, C^-
- C^+ and C^- are one-sided upper and lower cusums, respectively.

Upper and Lower CUSUMs

- Upper CUSUM

$$C_i^+ = \max\left[0, x_i - (\mu_0 + k) + C_{i-1}^+\right]$$

- Lower CUSUM

$$C_i^- = \max\left[0, (\mu_0 - k) - x_i + C_{i-1}^-\right]$$

Starting values are $C_0^+ = C_0^- = 0$

k is the **reference value** (or allowance or slack value)

If either statistic exceed a decision interval H, the process is considered to be out of control. Often taken as a $H = 5\sigma$

Selecting the Reference Value k

- The reference value k is often chosen halfway between the target μ_0 and the out-of-control value of the mean μ_1 that we are interested in detecting quickly.
- Shift is expressed in standard deviation units as $\mu_1 = \mu_0 + \delta\sigma$, then k is

$$k = \frac{\delta}{2}\sigma = \frac{|\mu_1 - \mu_0|}{2}$$

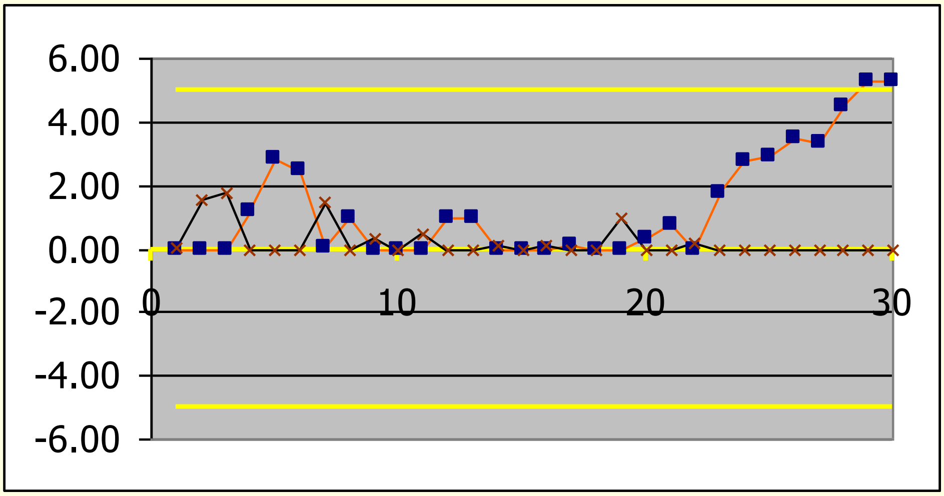
Example

- For the data of the last example, construct the upper and lower CUSUMs subject to the following
  - Interested in detecting shifts of the order of one standard deviation.
- From the above
  - Out-of-control value of the process mean is (10+1=11)
  - k=1/2 and H=5σ=5.
  - The equations for the statistics become

$$C_i^+ = \max\left[0, x_i - 10.5 + C_{i-1}^+\right]$$

$$C_i^- = \max\left[0, 9.5 - x_i + C_{i-1}^-\right]$$

N reports the number of consecutive nonzeros



The Standardized Cusums

- It may be of interest to standardize the variable x_i.

$$y_i = \frac{x_i - \mu_0}{\sigma}$$

- The standardized cusums are then

$$C_i^+ = \max\left[0, y_i - k + C_{i-1}^+\right]$$
$$C_i^- = \max\left[0, k - y_i + C_{i-1}^-\right]$$

- The cusum control chart indicates the process is out of control.
- The next step is to search for an assignable cause, take corrective action required, and reinitialize the cusum at zero.
- If an adjustment has to be made to the process, may be helpful to estimate the process mean following the shift.

Rational Subgroups

- Shewhart chart performance is improved with rational subgrouping.
- Cusum is not necessarily improved with rational subgrouping.
- Only if there is significant economy of scale or some other reason for taking larger samples should rational subgrouping be considered with the cusum.

Improving Cusum Responsiveness for Large Shifts

- Cusum control chart is not as effective in detecting large shifts in the process mean as the Shewhart chart.
- An alternative is to use a combined cusum-Shewhart procedure for on-line control.
- The combined cusum-Shewhart procedure can improve cusum responsiveness to large shifts.

The Fast Initial Response or Headstart Feature

- These procedures were introduced to increase sensitivity of the cusum control chart upon start-up.
- The fast initial response (FIR) or headstart sets the starting values C_0^+, C_0^- equal to some nonzero value, typically H/2.
- Setting the starting values to H/2 is called a 50 percent headstart.

One-Sided Cusums

- There are practical situations where a single one-sided cusum is useful.
- If a shift in only one direction is of interest then a one-sided cusum would be applicable.

A Cusum for Monitoring Process Variability

- Let $x_i \sim N(\mu_0, \sigma)$
- The standardized value of x_i is $y_i = (x_i - \mu_0)/\sigma$
- A new standardized quantity (Hawkins (1981) (1993)) is given by

$$v_i = \frac{\sqrt{|y_i|} - 0.822}{0.349}$$

- Hawkins suggest that the v_i are sensitive to variance changes rather than mean changes.

A Cusum for Monitoring Process Variability

- $\nu_I \sim N(0, 1)$, two one-sided standardized **scale cusums** are

$$S_i^+ = \max\left[0, \nu_i - k + S_{i-1}^+\right]$$
$$S_i^- = \max\left[0, \nu_i - k + S_{i-1}^-\right]$$

where $S_i^+ = S_i^- = 0$

if either statistic exceeds h, the process is considered out of control.

The V-Mask Procedure

- The V-mask procedure is an alternative to the tabular cusum.
- It is often strongly advised **not** to use the V-mask procedure for several reasons.
  1. The V-mask is a two-sided scheme; it is not very useful for one-sided process monitoring problems.
  2. The headstart feature, which is very useful in practice, cannot be implemented with the V-mask.
  3. It is sometimes difficult to determine how far backwards the arms of the V-mask should extend, thereby making interpretation difficult for the practitioner.
  4. Ambiguity associated with α and β.

The Exponentially Weighted Moving Average Control Chart (EWMA)

- The exponentially weighted moving average (EWMA) is defined as

$$z_i = \lambda x_i + (1-\lambda) z_{i-1}$$

- where

  $0 < \lambda \leq 1$ is a constant.

  $z_0 = \mu_0$ (sometimes $z_0 = \bar{x}$)

EWMA: Monitoring the Process Mean

- The control limits for the EWMA control chart are

$$UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2i}\right]}$$
$$CL = \mu_0$$
$$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2i}\right]}$$

- where L is the width of the control limits.

- As i gets larger, the term $[1- (1 - \lambda)^{2i}]$ approaches unity.
- This indicates that after the EWMA control chart has been running for several time periods, the control limits will approach **steady-state** values given by

$$UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}}$$

$$CL = \mu_0$$

$$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}}$$

EWMA Chart Design

- The design parameters of the chart are L and λ.
- The parameters can be chosen to give desired ARL performance.
- For λ
  - $0.05 \le \lambda \le 0.25$ works well in practice.
- *For L*
  - $L = 3$ works reasonably well especially with the larger value of λ.
  - L between 2.6 and 2.8 is useful when $\lambda \le 0.1$.
- Similar to the cusum, the EWMA performs well against *small shifts* but does not react to large shifts as quickly as the Shewhart chart.
- EWMA is often *superior* to the cusum for larger shifts particularly if $\lambda > 0.1$.
- the **individuals** control chart is sensitive to non-normality. A properly designed EWMA is less sensitive to the normality assumption.

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Example

- Apply EWMA for the data given in the last example with λ = 0.1 and L=2.7.

- Calculate the EWMA z_i from th egiven x_i. The following is the first point calculation

$$z_1 = \lambda x_1 + (1-\lambda) z_0$$
$$= 0.1(9.45) + 0.9(10) = 9.945$$

- The control limits are found using i=1,2,…. Notice that the UCL evolved to 10.62 and LCL evolved to 9.38. Any points outside these values are out-of-control. Clearly the last two point are out-of-control.