Numerical Methods

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Chapter (5): Bracketing Methods:

→ The Main point of the chapter:

This chapter on roots of equations deals with methods that exploit the fact that a function typically changes sign in the vicinity of a root. These techniques are called *bracketing methods* because two initial guesses for the root are required. As the name implies, these guesses must "bracket," or be on either side of, the root. The particular methods described herein employ different strategies to systematically reduce the width of the bracket and, hence, home in on the correct answer.

5.1: Graphical Methods:

A simple method for obtaining an estimate of the root of the equation $f(x) = 0$ is to make a plot of the function and observe where it crosses the x axis. This point, which represents the x value for which $f(x) = 0$, provides a rough approximation of the root.

EXAMPLE 5.1 The Graphical Approach

Problem Statement. Use the graphical approach to determine the drag coefficient c ? needed for a parachutist of mass $m = 68.1$ kg to have a velocity of 40 m/s after free-falling for time $t = 10$ s. *Note:* The acceleration due to gravity is 9.81 m/s^2.

→ Remember the equation of free falling object:

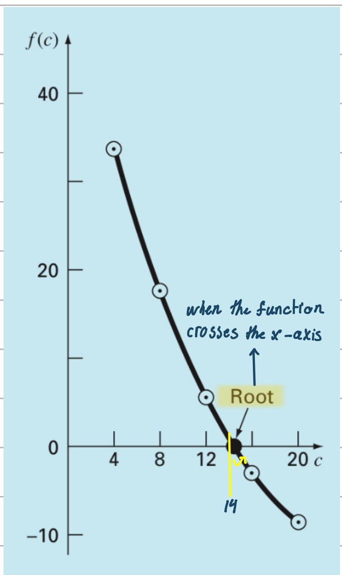
$$v(t) = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right)$$

$$40 = \frac{9.81(68.1)}{c}\left(1 - e^{-(c/68.1)10}\right)$$

$$f(c) = \frac{9.81(68.1)}{c}\left(1 - e^{-(c/68.1)10}\right) - 40$$

c	f(c)
4	34.190
8	17.712
12	6.114
16	−2.230
20	−8.368

– we substituted various values of (c) into the Equation ⇒ to find f(c)



– we plotted the Data in the above Table.

→ we can see that our Root falls Between the (12) and (16), closer to the (16), we can estimate that it's (14.7) for Example, How can we check out estimate ?

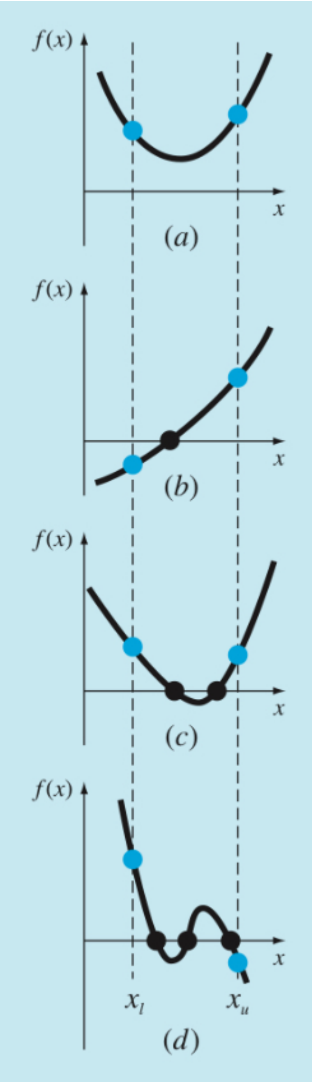
– using the original equation:

$$v = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right)$$

$$v = \frac{9.81(68.1)}{14.7}\left(1 - e^{-(14.7/68.1)10}\right)$$

$v = 40.1$ m/s → it's pretty close to (40 m/s) as mentioned in the question above

→ General way a Root May occur:



→ Both have the same sign ⇒ No sign change
— this means that the function either has No Root or even numbers of Roots.

→ The function changed sign
— which means we eithe have a single Root or odd Number of Roots.

→ the Sign of the overall didn't change
— so we either have No Root or even Number of Roots
(same as (a))

→ the function changed sign at the end
— so there will be an odd Number of Roots.

5.2: Bisection Method

When applying the graphical technique in Example 5.1, you have observed (Fig. 5.1) that $f(x)$ changed sign on opposite sides of the root. In general, if $f(x)$ is real and continuous in the interval from x_l to x_u and $f(x_l)$ and $f(x_u)$ have opposite signs, that is,

$$f(x_l)f(x_u) < 0 \tag{5.1}$$ → so one of them must have a (−) sign.

then there is at least one real root between x_l and x_u.

— steps of Bisection Method:

Step 1: Choose lower x_l and upper x_u guesses for the root such that the function changes sign over the interval. This can be checked by ensuring that $f(x_l)f(x_u) < 0$.

Step 2: An estimate of the root x_r is determined by

$$\boxed{x_r = \frac{x_l + x_u}{2}}$$

Step 3: Make the following evaluations to determine in which subinterval the root lies:
(a) If $f(x_l)f(x_r) < 0$, the root lies in the lower subinterval. Therefore, set $x_u = x_r$ and return to step 2.
(b) If $f(x_l)f(x_r) > 0$, the root lies in the upper subinterval. Therefore, set $x_l = x_r$ and return to step 2.
(c) If $f(x_l)f(x_r) = 0$, the root equals x_r; terminate the computation.

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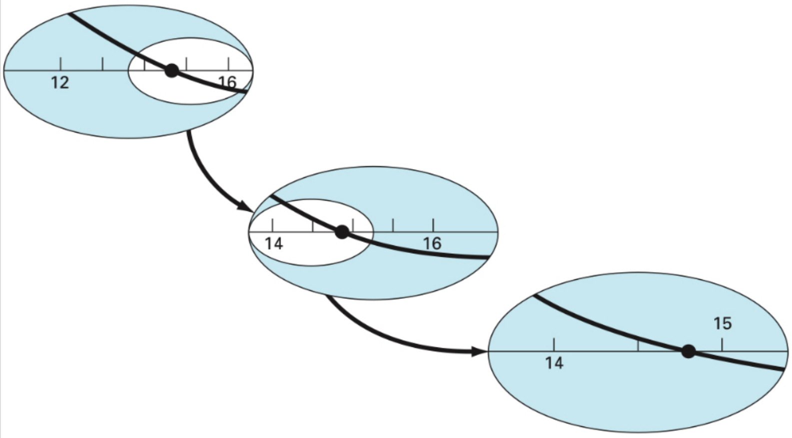


EXAMPLE 5.3 Bisection

Problem Statement. Use bisection to solve the same problem approached graphically in Example 5.1.

Solution. The first step in bisection is to guess two values of the unknown (in the present problem, c) that give values for $f(c)$ with different signs. From Fig. 5.1, we can see that the function changes sign between values of 12 and 16. Therefore, the initial estimate of the root x_r lies at the midpoint of the interval

$$x_r = \frac{\overset{x_L}{12} + \overset{x_U}{16}}{2} = 14$$

This estimate represents a true percent relative error of $\varepsilon_t = 5.3\%$ (note that the true value of the root is 14.8011). Next we compute the product of the function value at the lower bound and at the midpoint:

$$f(12)f(14) = 6.114(1.611) = 9.850 > 0 \Rightarrow \text{so } x_r = x_L$$

which is greater than zero, and hence no sign change occurs between the lower bound and the midpoint. Consequently, the root must be located between 14 and 16. Therefore, we create a new interval by redefining the lower bound as 14 and determining a revised root estimate as

$$x_r = \frac{14 + 16}{2} = 15$$

which represents a true percent error of $\varepsilon_t = 1.3\%$. The process can be repeated to obtain refined estimates. For example,

$$f(14)f(15) = 1.611(-0.384) = -0.619 < 0 \Rightarrow \text{so } x_r = x_U$$

Therefore, the root is between 14 and 15. The upper bound is redefined as 15, and the root estimate for the third iteration is calculated as

$$x_r = \frac{14 + 15}{2} = 14.5$$

which represents a percent relative error of $\varepsilon_t = 2.0\%$. The method can be repeated until the result is accurate enough to satisfy your needs.

Note : the True error Doesn't necessarily Decrease with each itiration. ⇒ so we need a stopping ciriteria.

- Bcs of the error problem we faced in the previous question we need a new way to calculate the error for this Method.

$$\varepsilon_a = \left| \frac{x_r^{\text{new}} - x_r^{\text{old}}}{x_r^{\text{new}}} \right| 100\%$$

→ we use this Absolute value Bcs we're only concerned with the Magnitude Not the sign

- x_r^{new} : the current iteration
- x_r^{old} : the previous iteration

EXAMPLE 5.4 Error Estimates for Bisection

Problem Statement. Continue Example 5.3 until the approximate error falls below a stopping criterion of $\varepsilon_s = 0.5\%$. Use Eq. (5.2) to compute the errors.

Solution. The results of the first two iterations for Example 5.3 were 14 and 15. Substituting these values into Eq. (5.2) yields

$$|\varepsilon_a| = \left| \frac{15 - 14}{15} \right| 100\% = 6.667\%$$

Recall that the true percent relative error for the root estimate of 15 was 1.3%. Therefore, ε_a is greater than ε_t. This behavior is manifested for the other iterations:

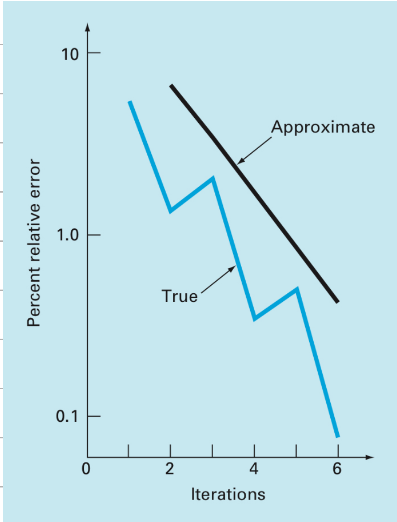
Iteration	x_l	x_u	x_r	ε_a (%)	ε_t (%)
1	12	16	14		5.413
2	14	16	15	6.667	1.344
3	14	15	14.5	3.448	2.035
4	14.5	15	14.75	1.695	0.345
5	14.75	15	14.875	0.840	0.499
6	14.75	14.875	14.8125	0.422	0.077

Decreasing

we can tell that the True error isn't necessarily going down

Thus, after six iterations ε_a finally falls below $\varepsilon_s = 0.5\%$, and the computation can be terminated.

These results are summarized in Fig. 5.7. The "ragged" nature of the true error is due to the fact that, for bisection, the true root can lie anywhere within the bracketing interval. The true and approximate errors are far apart when the interval happens to be centered on the true root. They are close when the true root falls at either end of the interval.



- The Approximate error Doesn't provide an exact estimate
- (ε_a) is always greater than (ε_t)

→ However, Bisection Method is slower than other Methods.

when ε_a falls below ε_s, the computation could be terminated with confidence that the root is known to be at least as accurate as the prespecified acceptable level.

Although it is always dangerous to draw general conclusions from a single example, it can be demonstrated that ε_a will always be greater than ε_t for the bisection method. This is because each time an approximate root is located using bisection as $x_r = (x_l + x_u)/2$, we know that the true root lies somewhere within an interval of $(x_u - x_l)/2 = \Delta x/2$. Therefore, the root must lie within $\pm\Delta x/2$ of our estimate (Fig. 5.8). For instance, when Example 5.3 was terminated, we could make the definitive statement that

$$x_r = 14.5 \pm 0.5$$

Because $\Delta x/2 = x_r^{\text{new}} - x_r^{\text{old}}$ (Fig. 5.9), Eq. (5.2) provides an exact upper bound on the true error. For this bound to be exceeded, the true root would have to fall outside the bracketing interval, which, by definition, could never occur for the bisection method. As illustrated in a subsequent example (Example 5.6), other root-locating techniques do not always behave as nicely. Although bisection is generally slower than other methods,

FIGURE 5.8
Three ways in which the interval may bracket the root. In (a) the true value lies at the center of the interval, whereas in (b) and (c) the true value lies near the extreme. Notice that the discrepancy between the true value and the midpoint of the interval never exceeds half the interval length, or $\Delta x/2$.

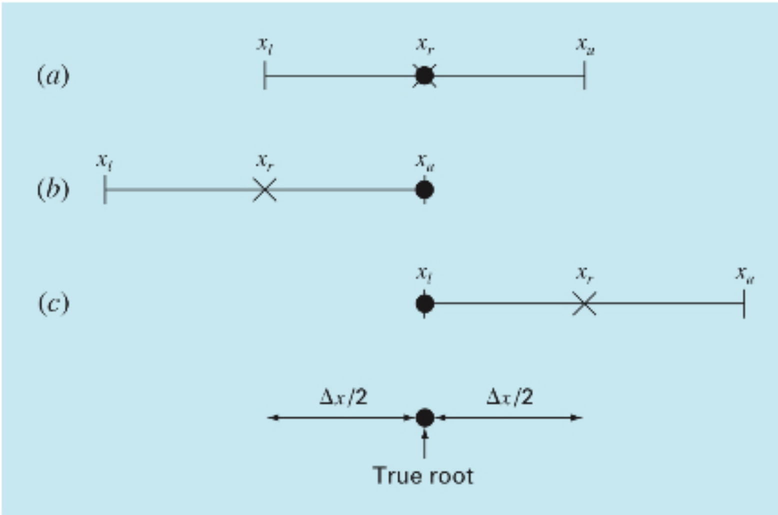
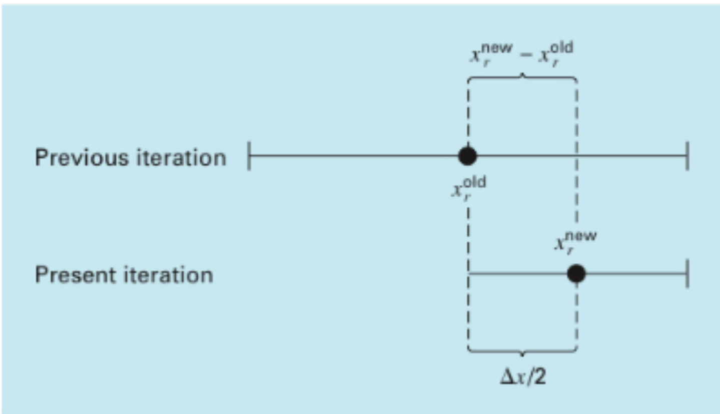
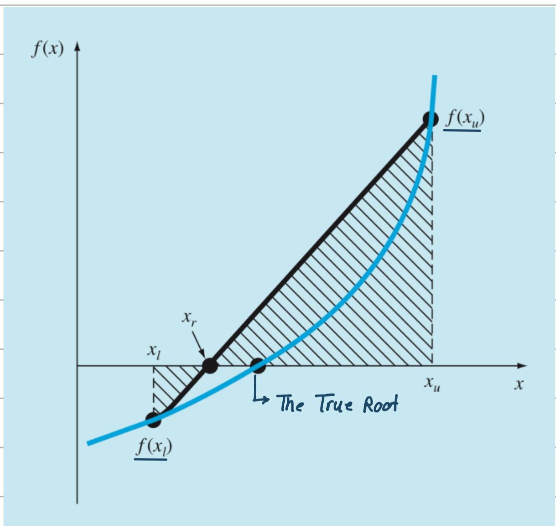


FIGURE 5.9
Graphical depiction of why the error estimate for bisection ($\Delta x/2$) is equivalent to the root estimate for the present iteration (x_r^{new}) minus the root estimate for the previous iteration (x_r^{old}).



5.3: The false position Method:

- false position is an Alternative for the Bisection Method Based on graphical insight.
- we consider the Magnitude of $f(x_u)$ and $f(x_L)$.
- we also join $f(x_u)$ and $f(x_L)$ with a line, and the intersection of that line with the x-axis Represents a Better estimate of the Root
- it's also called "linear interpolation Method".



$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

↳ the other steps are the same as the Bisection Method.

EXAMPLE 5.5 False Position

Problem Statement. Use the false-position method to determine the root of the same equation investigated in Example 5.1 [Eq. (E5.1.1)].

Solution. As in Example 5.3, initiate the computation with guesses of $x_l = 12$ and $x_u = 16$.

First iteration:

← عوضت في هاي المعادلة :

$$\boxed{f(c) = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right) - v(t)}$$

$$x_l = 12 \qquad f(x_l) = 6.1139$$
$$x_u = 16 \qquad f(x_u) = -2.2303$$
$$x_r = 16 - \frac{-2.2303(12 - 16)}{6.1139 - (-2.2303)} = 14.309$$

which has a true relative error of 0.88 percent.

Second iteration:

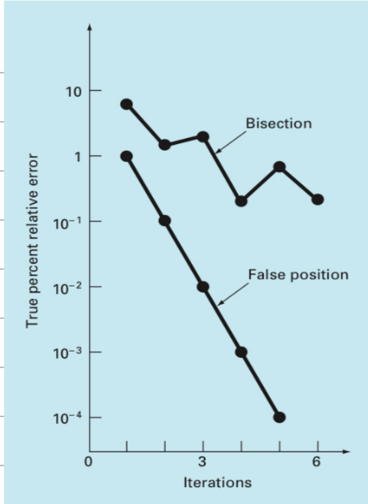
$$f(x_l)f(x_r) = -1.5376 < 0 \Rightarrow x_r = x_u$$

Therefore, the root lies in the first subinterval, and x_r becomes the upper limit for the next iteration, $x_u = 14.9113$:

$$x_l = 12 \qquad f(x_l) = 6.1139$$
$$x_u = 14.9309 \qquad f(x_u) = -0.2515$$
$$x_r = 14.9309 - \frac{-0.2515(12 - 14.9309)}{6.1139 - (-0.2515)} = 14.8151$$

which has true and approximate relative errors of 0.09 and 0.78 percent. Additional iterations can be performed to refine the estimate of the roots.

← نفس الي كتت أعله في ال Bisection بس صرت احسب ال (x_r) بطاريقة حيز باخذ فيها في عين الاعتبار قيمة $f(x_u)$, $f(x_L)$.



→ We can see that the E_a for the false position Method decreases faster than the Bisection (which helps me to get to the Root faster)

Chapter 6 : open Methods:

- Remember in Bracketing Methods, The True Root is Between the upper and Lower limit these Methods are called Convergent (They're getting close to the Real value)
- open Methods Do not Consider upper and Lower limits, we instead Require a single or two starting value.
- open Methods can converge or Diverge, But they converge faster than the Bracketing Methods.

6.1: simple Fixed-point iteration:

- we rearrange the function such that the left side of the equation has an "x"

For Ex :

1. $\sin x = 0 \Rightarrow \sin x + x = x$
   $+x \quad +x$
2. $x^2 - 2x + 3 = 0 \Rightarrow \frac{x^2+3}{2} = x$

- so what we do is provide a formula for x Based on an old value of x

$$\underbrace{x_{i+1}}_{\text{New}} = g(\underbrace{x_i}_{\text{previous}})$$

> • Remember:
> - we calculate the App. error:
>
> $$\varepsilon_a = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| 100\%$$

EXAMPLE 6.1 Simple Fixed-Point Iteration

Problem Statement. Use simple fixed-point iteration to locate the root of $f(x) = e^{-x} - x$.

$f(x) = e^{-x} - x \Rightarrow$

$x = e^{-x}$

i	x	$\varepsilon_a \%$	$\varepsilon_t \%$
0	0	—	100.0
1	e^{-1}	100.0	
2	e^{-2}	$\frac{e^{-2}-e^{-1}}{e^{-2}}$	
3	e^{-3}	⋮	
4	⋮		

current iteration - previous.

↳ I keep going on until I reach an App error that's less than Tolerance.

→ convergence:

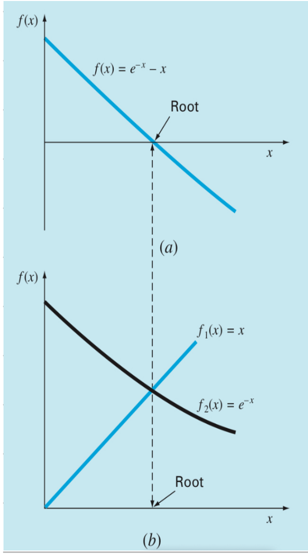
- The Relative error of each iteration is proportional By a factor ⇒ linear convergance
- There's a graphical Alternative Approach ⟶ two-carved Method

$f_1(x) = f_2(x)$

↳ $y_1 = f_1(x)$
↳ $y_2 = f_2(x)$ } → we plot the two functions, and their intersection is the Root.

EXAMPLE 6.2 The Two-Curve Graphical Method

Problem Statement. Separate the equation $e^{-x} - x = 0$ into two parts and determine its root graphically.



Solution. Reformulate the equation as $y_1 = x$ and $y_2 = e^{-x}$. The following values can be computed:

x	y_1	y_2
0.0	0.0	1.000
0.2	0.2	0.819
0.4	0.4	0.670
0.6	0.6	0.549
0.8	0.8	0.449
1.0	1.0	0.368

These points are plotted in Fig. 6.2*b*. The intersection of the two curves indicates a root estimate of approximately $x = 0.57$, which corresponds to the point where the single curve in Fig. 6.2*a* crosses the x axis.

6.2: Newton's Raphson Method:

- when the initial guess at the Root is x_i, A Tangent line is extended from that point, where this line intersects with the x-axis, and this gives an improved estimate of the Root.

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Modified estimate   previous estimate

↙

EXAMPLE 6.3 Newton-Raphson Method

Problem Statement. Use the Newton-Raphson method to estimate the root of $f(x) = e^{-x} - x$, employing an initial guess of $x_0 = 0$.

$x_i = e^{x_i}$

$$x_{i+1} = x_i - \frac{e^{-x_i} - x_i}{e^{-x_i} - 1}$$

i	x_i	$\varepsilon_t\%$
0	0	
1	e^{-1}	
2	e^{-2}	
3	e^{-3}	
⋮	⋮	

$f(x) = e^{-x_i} - x$

$f'(x) = e^{-x_i} - 1$

→ Error Analysis of Newton's-Raphson :

- we can predict the next error using this equation:

$$E_{t,i+1} = \frac{-f''(x_r)}{2f'(x_r)} E_{t,i}$$

x_{i+1}

previous error

new error

Note:

→ There's No certain Criteria of Convergence for Newtons Raphson Method, it converges Based on the function Nature.

6.3: The secant Methods:

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}}$$

EXAMPLE 6.6 The Secant Method

Problem Statement. Use the secant method to estimate the root of $f(x) = e^{-x} - x$. Start with initial estimates of $x_{-1} = 0$ and $x_0 = 1.0$.

$f(x) = e^{-x} - x$

$f'(x) = e^{-x} - 1$

1st itiration:

$x_{i-1} = 0$

$x_i = 1.0$

$$x_{i+1} = 1 - \frac{(-0.63212)(0-1)}{1-(-0.63212)}$$

$\boxed{x_{i+1} = 0.6127}$

2nd itiration:

$x_{i-1} = 1.0$

$x_i = 0.6127$

$$x_{i+1} = 0.6127 - \frac{-0.07081(1-0.61270)}{-0.63212-(-0.07081)}$$

$\boxed{x_{i+1} = 0.56384}$

$$\Rightarrow \varepsilon_a = \left| \frac{0.56384 - 0.6127}{0.56384} \right| = 0.086$$

Chapter (9): Gauss Elimination:

→ this is how we generally Represent an Equation:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

→ a and b are constants

9.1 : solving small Number of equations:

→ Methods:

1. The graphical Method:

- we plot them on cartissian coordinates (x, y), one corresponding to x_1 and one for x_2
- since we're dealing with "linear systems", the equations are straight lines.

$a_{11}x_1 + a_{12}x_2 = b_1 \Rightarrow x_1 = (-a_{12}x_1 - b_1)/a_{11}$

$a_{21}x_1 + a_{22}x_2 = b_2 \Rightarrow x_2 = (-a_{21}x_1 - b_2)/a_{22}$

EXAMPLE 9.1 The Graphical Method for Two Equations

Problem Statement. Use the graphical method to solve

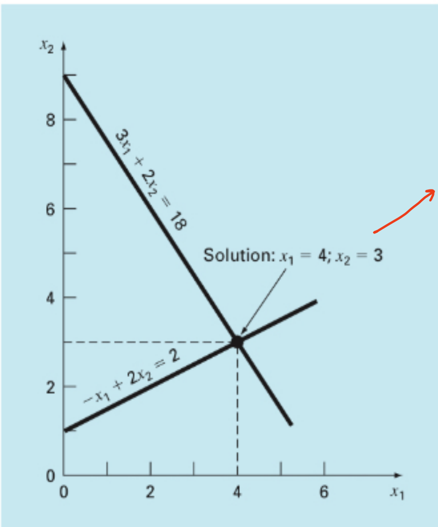
$$3x_1 + 2x_2 = 18 \quad \text{(E9.1.1)}$$
$$-x_1 + 2x_2 = 2 \quad \text{(E9.1.2)}$$

Solution. Let x_1 be the abscissa. Solve Eq. (E9.1.1) for x_2:

$$x_2 = -\frac{3}{2}x_1 + 9$$

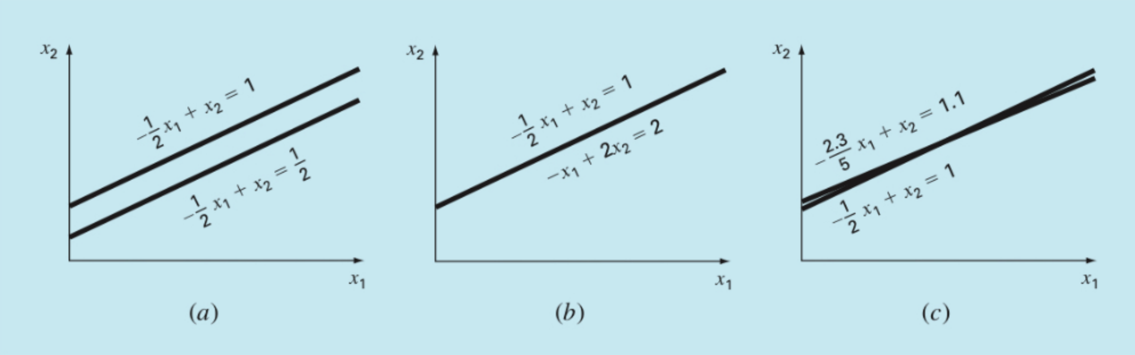
which, when plotted on Fig. 9.1, is a straight line with an intercept of 9 and a slope of $-3/2$.

FIGURE 9.1
Graphical solution of a set of two simultaneous linear algebraic equations. The intersection of the lines represents the solution.



to check these values we can substitute them in the original Equations

→ But this Method Breaks Beyond 3 equations, and we can face some problems like:



(a) ⇒ the two lines are parallel ⇒ shows No solution

(B) ⇒ the two lines are coincident (متطابقين) ⇒ infinite Number of solutions

(C) ⇒ the two lines are so close to being coincident ⇒ Extremely hard to Determine solution.

↪ those system are call "ill-conditioned system"

2. Cramer's Rule:

- suits a small Number of solutions:

$[A]\{x\} = \{B\}$

↳ coefficient of Matrix ⇒ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

↪ Determinant: $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

- 2nd order Determinant:

$D = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \boxed{D = a_{11}a_{22} - a_{12}a_{21}}$

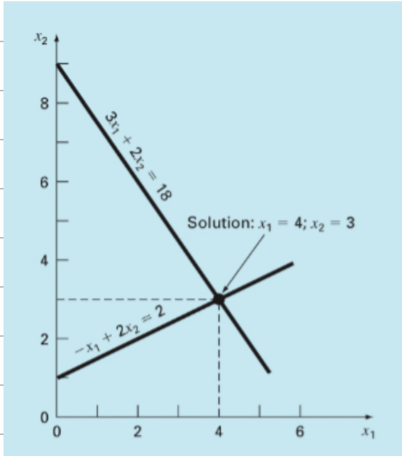
- 3rd order Determinant:

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

EXAMPLE 9.2 Determinants

Problem Statement. Compute values for the determinants of the systems represented in Figs. 9.1 and 9.2.

1.

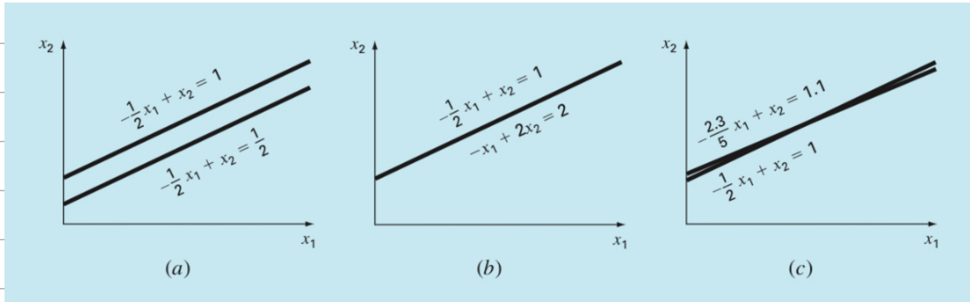


Solution:

1. $\left.\begin{aligned} 3x_1 + 2x_2 &= 18 \\ -x_1 + 2x_2 &= 2 \end{aligned}\right]$

$D = \begin{vmatrix} 3 & 2 \\ -1 & 2 \end{vmatrix} = 3(2) - 2(-1) = 8$

2.



(a) (b) (c)

a) $D = \begin{vmatrix} -1/2 & 1 \\ -1/2 & 1 \end{vmatrix}$ b) $D = \begin{vmatrix} -1/2 & 1 \\ -1 & 2 \end{vmatrix}$ c) $D = \begin{vmatrix} 2.3/5 & 1 \\ -1/2 & 1 \end{vmatrix}$

$D = (-1/2) - (-1/2)$ $D = -1/2(2) - (-1)$ $D = 2.3/5 - (-1/2)$

$= 0$ $= 0$ $= -0.04$

↪ (a) and (b) are singular systems, the had zero Determinants, while c had almost a zero Determinant, so the system is almost singular.

• Cramer's Rule:

- each unknown in the system will Be expressed in terms of two Determinants

$$\boxed{x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D}}$$

→ we Basically Replaced the columns of coefficients for the unknown with $b_1, b_2, b_3, \ldots$

↳ same thing for $x_2, x_3, \ldots$

EXAMPLE 9.3 Cramer's Rule

Problem Statement. Use Cramer's rule to solve

$$0.3x_1 + 0.52x_2 + x_3 = -0.01$$
$$0.5x_1 + x_2 + 1.9x_3 = 0.67$$
$$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44$$

Solution :

$$\begin{bmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{bmatrix} \Rightarrow D = 0.3 \begin{vmatrix} 1 & 1.9 \\ 0.3 & 0.5 \end{vmatrix} - 0.5 \begin{vmatrix} 0.52 & 1 \\ 0.3 & 0.5 \end{vmatrix} + 0.1 \begin{vmatrix} 0.52 & 1 \\ 1 & 1.9 \end{vmatrix}$$

$$D = 0.3(0.5 - 0.3(1.9)) + 0.5((0.52)(0.5) - 0.3) + 0.1((1.9)(0.52) - 1)$$

$$D = -0.0022$$

1.

$$Dx_1 = \begin{vmatrix} -0.01 & 0.52 & 1 \\ 0.67 & 1 & 1.9 \\ -0.44 & 0.3 & 0.5 \end{vmatrix} = -0.01 \begin{vmatrix} 1 & 1.9 \\ 0.3 & 0.5 \end{vmatrix} - 0.67 \begin{vmatrix} 0.52 & 1 \\ 0.3 & 0.5 \end{vmatrix} + -0.44 \begin{vmatrix} 0.52 & 1 \\ 1 & 1.9 \end{vmatrix}$$

$$= -0.01\,(0.5 - 1.9(0.3)) - 0.67\,(0.52(0.5) - 0.3) - 0.44\,(0.52(1.9 - 1)$$

$$= 0.03278$$

$$Dx_2 = \begin{vmatrix} 0.3 & -0.01 & 1 \\ 0.5 & 0.67 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{vmatrix} = 0.0649$$

$$Dx_3 = \begin{vmatrix} 0.3 & 0.52 & -0.01 \\ 0.5 & 1 & 0.67 \\ 0.1 & 0.3 & 0.3 \end{vmatrix} = -0.04356$$

2.

$$x_1 = \frac{Dx_1}{D} \qquad x_2 = \frac{Dx_2}{D} \qquad x_3 = \frac{Dx_3}{D}$$

$$x_1 = \frac{0.0327}{-0.0022} \qquad x_2 = \frac{0.0649}{-0.0022} \qquad x_3 = \frac{-0.04356}{-0.0022}$$

$$x_1 = -14.9 \qquad x_2 = -29.5 \qquad x_3 = -19.8$$

↪ But carmer's Rule Becomes impractical as number of unknowns increases (Number of equations increases)

3. elimination of unknowns:

- we Multiply one of the equations By constants, so we can eliminate one of the unknowns.

$$\left(a_{11}x_1 + a_{12}x_2 = b_1\right) \times \frac{a_{21}}{a_{11}} \Rightarrow a_{21}x_1 + a_{12}\left(\frac{a_{21}}{a_{11}}\right)x_2 = b_1\left(\frac{a_{21}}{a_{11}}\right)$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

(we can eleminate (x_1))

$$x_1 = \frac{\overbrace{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}^{D x_1}}{\underbrace{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}_{D}} = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}}$$

☆ same as Cramer's Rule ☆

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

EXAMPLE 9.4 Elimination of Unknowns

Problem Statement. Use the elimination of unknowns to solve (recall Example 9.1)

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2$$

solution:

$$\left[\begin{array}{cc:c} 3 & 2 & 18 \\ -1 & 2 & 2 \end{array}\right]$$

1. $D = (3)(2) - (-2)$

$D = 8$

2. $x_1 = \frac{18(2) - 2(2)}{8} = 4$

$x_2 = \frac{2(3) - 18(-1)}{8} = 3$

9.2: Naive Gauss elimination:

- This Technique Consists of two phases:

1. Forward elimination
2. Back substitution

1. Forword elimination of Unknowns:

- the point is to reduce the set of equations to an upper Triangular system.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \quad \rightarrow \text{pivot Row}$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n$$

→ to eliminate the first unknown (x_1) from the 2nd equation, we'll Multiply the 1st equation By (a_{21}/a_{11}), which Results in:

$$\boxed{a_{21}x_1 + \left(\frac{a_{21}}{a_{11}}\right)a_{12}x_2 + \cdots \left(\frac{a_{21}}{a_{11}}\right)a_{1n}x_n = \left(\frac{a_{21}}{a_{11}}\right)b_1}$$

⇒ pivot equation

→ Number of Row.

Note: (a_{21}/a_{11}) is the pivot Coeff.

- Now we substract the 1st equation from the 2nd Equation, which gives:

$$\underbrace{\left(a_{22} - \left(\frac{a_{21}}{a_{11}}\right)a_{12}\right)}_{a_{22}'} x_2 + \cdots \underbrace{\left(a_{2n} - \left(\frac{a_{21}}{a_{11}}\right)a_{1n}\right)}_{a_{2n}'} x_n = \underbrace{b_2 - \left(\frac{a_{21}}{a_{11}}\right)b_1}_{b_2'}$$

→ the prime (′) means that the original values are changed.

Augmented Matrix.

$$\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$$\Downarrow$$

$$\begin{array}{l} \\ \text{1st substitution} \leftarrow \\ \text{2nd}\ \text{''} \leftarrow \end{array}\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ \_ & a_{22}' & a_{23}' & b_2' \\ \_ & \_ & a_{33}'' & b_3'' \end{array}\right]$$

Forward elimination

$$\Downarrow$$

$$x_3 = b_3''/a_{33}''$$
$$x_2 = (b_2' - a_{23}'x_3)/a_{22}'$$
$$x_1 = (b_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$$

Back substitution

2. Back substitution:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad \text{find } x_1$$
$$a_{22}'x_2 + a_{23}'x_3 = b_2'' \quad \text{find } x_2$$
$$a_{33}''x_3 = b_3'' \quad \text{1. Find } x_3$$

Final Result

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{22}'x_2 + a_{23}'x_3 + \cdots + a_{2n}'x_n &= b_2' \\ a_{33}''x_3 + \cdots + a_{3n}''x_n &= b_3'' \\ \vdots \qquad &\vdots \\ a_{n3}''x_3 + \cdots + a_{nn}''x_n &= b_n'' \end{aligned}}$$

EXAMPLE 9.5 Naive Gauss Elimination

Problem Statement. Use Gauss elimination to solve

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \quad (E9.5.1)$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \quad (E9.5.2)$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4 \quad (E9.5.3)$$

Carry six significant figures during the computation.

It's always Better to use more sig figures to avoid the effect of Round off errors

Solution:

$$\left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & 0.3 & -19.3 \\ 0.3 & 0.2 & 10 & 71.4 \end{array}\right]$$

$\left(\frac{0.1}{3}\right) \Rightarrow 0.1 \quad -0.003333 \quad 0.066666$

$\left(\frac{0.3}{3}\right) \Rightarrow 0.3 \quad 0.700000 \quad 0.0300000$

$\Downarrow$

$$\left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0 & 7.00333 & 0.293333 & -19.5617 \\ 0 & -0.190000 & 10.0200 & 70.6150 \end{array}\right]$$

$\rightarrow \left(\frac{-0.190000}{7.00333}\right) \Rightarrow 0 \quad 0.190000 \quad 0.007958$

$\Downarrow$

$$\left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0 & 7.00333 & -0.293333 & -19.5617 \\ 0 & 0 & 10.0120 & 70.0843 \end{array}\right]$$

$\rightarrow 10.0120x_3 = 70.0843$

$\boxed{x_3 = 7.00003}$

(1) $7.00333x_2 - 0.293333(7.00003) = -19.5617$

$\boxed{x_2 = -2.50000}$

(3) $3x_1 - 0.1(-2.50000) - 0.2(7.00003) = 7.85$

$\boxed{x_1 = 3.00000}$

9.3 : Pitfalls of elimination Method:

1. Dividing By Zero:

for example:

$2x_2 + 3x_3 = 8$ $\rightarrow$ $x_1 = $ zero , so in this case we would Be deviding By zero

$4x_1 + 6x_2 + 7x_3 = -3$ since $a_{11} =$ zero.

$2x_1 + x_2 + 6x_3 = 5$

2. Round - off errors:

- this problem gets importantly sig when large numbers of equations are to be solved.
- using more significant figures can lessen this problem.

3. ill-conditioned system:

- that has a determinant that's almost a zero.
- small changes in coefficients can lead to large changes in solution.

EXAMPLE 9.7 Effect of Scale on the Determinant

Problem Statement. Evaluate the determinant of the following systems:

(a) From Example 9.1: → well-conditioned

$$3x_1 + 2x_2 = 18 \quad \text{(E9.7.1)}$$
$$-x_1 + 2x_2 = 2 \quad \text{(E9.7.2)}$$

(b) From Example 9.6: → ill-conditioned

$$x_1 + 2x_2 = 10 \quad \text{(E9.7.3)}$$
$$1.1x_1 + 2x_2 = 10.4 \quad \text{(E9.7.4)}$$

(c) Repeat (b) but with the equations multiplied by 10.

solution:

(a) $\begin{vmatrix} 3 & 2 \\ -1 & 2 \end{vmatrix} \Rightarrow D = 3(2) - 2(-1) = 8$

(b) $\begin{vmatrix} 1 & 2 \\ 1.1 & 2 \end{vmatrix} \Rightarrow D = 2 - 2(1.1) = -0.2$

⇒ There's a huge change

(c) $\begin{vmatrix} 10 & 20 \\ 11 & 20 \end{vmatrix} \Rightarrow D = 20(10) - 20(11) = -20$

↪ Multiplying the equation By a constant, Doesn't effect the solution, But it has a huge effect on the Determinant in ill-conditioned systems.

■ so what's the solution for these issues?

9.4: Techniques of improving the solution:

1. use of more sig figure

↳ explained Before

2. pivoting:

- Remember, real problems occur when the pivot element is zero, Bcs then we'll have to Divide By Zero in the normalization step.
- also if the pivot element's Magnitude is smaller than the other elements ⇒ causes Round-off errors.

↪ to solve these problems we use "partial pivoting"

- Rows are switched so that the largest number of Row is the pivot.

Keep in mind: Determinant Keeps its value, But its sign change with changing of Rows.

- تغيير بعدد فردي => بتتغير الاشارة.
- تغيير بعدد زوجي => ما بتتغير الاشارة.

Example (Not in Book, solved in class):

- solve this system of equations using gauss elimination

$0.0004x_1 + 1.402x_2 = 1.406$

$0.4003x_1 - 1.502x_2 = 2.501$

- first without pivoting:

$$\left[\begin{array}{cc|c} 0.0004 & 1.402 & 1.406 \\ 0.4003 & -1.502 & 2.501 \end{array}\right] \to *\left(\frac{0.4003}{0.0004}\right) \Rightarrow 0.4003 \quad 1403.05 \;\vdots\; 1407.05$$

$$\Downarrow$$

$$\left[\begin{array}{cc|c} 0.0004 & 1.402 & 1.406 \\ 0 & -1405 & -1404 \end{array}\right]$$

1) $-1405x_2 = -1404 \Rightarrow \boxed{x_2 = 0.9993}$

2) $0.0004x_1 + 1.402(0.9993) = 1.406 \Rightarrow \boxed{x_1 = 12.5}$

→ when we substitute these solutions in the original equation, they satisfy the 1st one But Not the 2nd one, Bcs of Round off errors, and the reason for that is that the pivots' Magnitude is so small compared to the other element.

- with partial pivoting:

$$\left[\begin{array}{cc|c} 0.4003 & -1.502 & 2.501 \\ 0.0004 & 1.402 & 1.406 \end{array}\right] \to *\left(\frac{0.0004}{0.4003}\right) \Rightarrow 0.0004 \quad -0.0015 \;\vdots\; 0.0025$$

$$\Downarrow$$

$$\left[\begin{array}{cc|c} 0.4003 & -1.502 & 2.501 \\ 0 & 1.4035 & 1.404 \end{array}\right]$$

1) $1.4035x_2 = 1.404 \Rightarrow \boxed{x_2 = 1.0004}$

$0.4003x_1 - 1.502(1.0004) = 2.501 \Rightarrow \boxed{x_1 = 10.001}$

↪ the two solutions satisfies Both equations ✓

3. Scaling:

- it minimize the Round-off errors for those cases where where some of the equations have larger coefficiants than others.
- standarize the size of Determinant (mentioned Before in ill-conditioned system)

EXAMPLE 9.10 Effect of Scaling on Pivoting and Round-Off

Problem Statement.

(a) Solve the following set of equations using Gauss elimination and a pivoting strategy:

$$2x_1 + 100{,}000x_2 = 100{,}000$$
$$x_1 + x_2 = 2$$

(b) Repeat the solution after scaling the equations so that the maximum coefficient in each row is 1.

(c) Finally, use the scaled coefficients to determine whether pivoting is necessary. However, actually solve the equations with the original coefficient values. For all cases, retain only three significant figures. Note that the correct answers are $x_1 = 1.00002$ and $x_2 = 0.99998$ or, for three significant figures, $x_1 = x_2 = 1.00$.

solution:

(a) $\left[\begin{array}{cc|c} 2 & 100{,}000 & 100{,}000 \\ 1 & 1 & 2 \end{array}\right] \rightarrow * \left(\frac{1}{2}\right) \Rightarrow 1 \quad 50{,}000 \mid 50{,}000$

$\Downarrow$

$\left[\begin{array}{cc|c} 2 & 100{,}000 & 100{,}000 \\ 0 & -49{,}999 & -49{,}998 \end{array}\right]$

$x_2 = (-49{,}998/-49{,}999) = \boxed{0.999} \simeq 1$

$x_1 = (100{,}000 - 100{,}000(1)/2 = 0$

! Round - off error!

(b) $\left[\begin{array}{cc|c} 0.0002 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right]$

$\downarrow$

$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0.0002 & 1 & 1 \end{array}\right] \rightarrow * \frac{0.0002}{1} \Rightarrow 0.0002 \quad 0.0002 \mid 0.0004$

$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0.9998 & 0.9996 \end{array}\right]$

$x_2 = 0.9996/0.9998 \simeq 1$

$x_1 = 2 - 1 = 1$

→ scaling and partial pivoting lead to a correct answer. (Not entirely, But Definitely decreased the error)

9.7: Gauss - Jordan:

Example: (in class)

- use gauss - Jordan to solve this system:

$x_1 + x_2 - x_3 = -3$

$6x_1 + 2x_2 + 2x_3 = 2$

$-3x_1 + 4x_2 + x_3 = 1$

$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 6 & 2 & 2 & 2 \\ -3 & 4 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 6 & 2 & 2 & 2 \\ 1 & 1 & -1 & -3 \\ -3 & 4 & 1 & 1 \end{array}\right] \rightarrow \div 6$ (Normalization)

$$\left[\begin{array}{ccc|c} 1 & 1/3 & 1/3 & 1/3 \\ 1 & 1 & -1 & -3 \\ -3 & 4 & 1 & 1 \end{array}\right] \begin{array}{l} \rightarrow *\left(\frac{1}{1}\right) \Rightarrow 1 \quad 1/3 \quad 1/3 \; \vdots \; 1/3 \\ \rightarrow *\left(\frac{-3}{1}\right) \Rightarrow -3 \quad -1 \quad -1 \; \vdots \; -1 \end{array}$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1/3 & 1/3 & 1/3 \\ 0 & 2/3 & 2/3 & -10/3 \\ 0 & 5 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1/3 & 1/3 & 1/3 \\ 0 & 5 & 2 & 2 \\ 0 & 2/3 & 2/3 & -10/3 \end{array}\right] \rightarrow \div 5 \text{ (Normalization)}$$

$$\left[\begin{array}{ccc|c} 1 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 2/5 & 2/5 \\ 0 & 2/3 & 2/3 & -10/3 \end{array}\right] \begin{array}{l} \rightarrow *\left(\frac{1}{3}\right) \Rightarrow 0 \quad 1/3 \quad 2/15 \; \vdots \; 2/15 \\ \rightarrow *\left(\frac{2}{3}\right) \Rightarrow 0 \quad 2/3 \quad 4/15 \; \vdots \; 4/15 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/5 & 1/5 \\ 0 & 1 & 2/5 & 2/5 \\ 0 & 0 & 2/5 & -54/15 \end{array}\right] \rightarrow \div(2/5) \text{ Normalization}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/5 & 1/5 \\ 0 & 1 & 2/5 & 2/5 \\ 0 & 0 & 1 & 9/4 \end{array}\right] \begin{array}{l} \rightarrow *\left(\frac{1}{5}\right) \Rightarrow 0 \quad 0 \quad 1/5 \; \vdots \; 9/20 \\ \rightarrow *\left(\frac{2}{5}\right) \Rightarrow 0 \quad 0 \quad 2/5 \; \vdots \; 9/10 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/4 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 9/4 \end{array}\right] \begin{array}{l} \rightarrow x_1 \\ \rightarrow x_2 \\ \rightarrow x_3 \end{array}$$

$\Rightarrow$ this Matrix is called identity Matrix

↳ also check example (9.12) in Book

Chapter (10):

10.1: LU Decomposition:

L → lower, U → Upper

$[A]\{x\} = \{B\}$

→ Gauss elemination Becomes innifficient when we're Dealing with Different coefficients of (B). so we use other Methods.

- What's the LU Decomposition?

A Method that seperates the determination of the Matrix [A] from the Right hand side [B]

بحيث انو تقدر أعدّل على قيم ال (B) من دونما أضطر أعيد الحل من أول وجديد.

$[A]\{x\} - \{B\} = 0$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$$

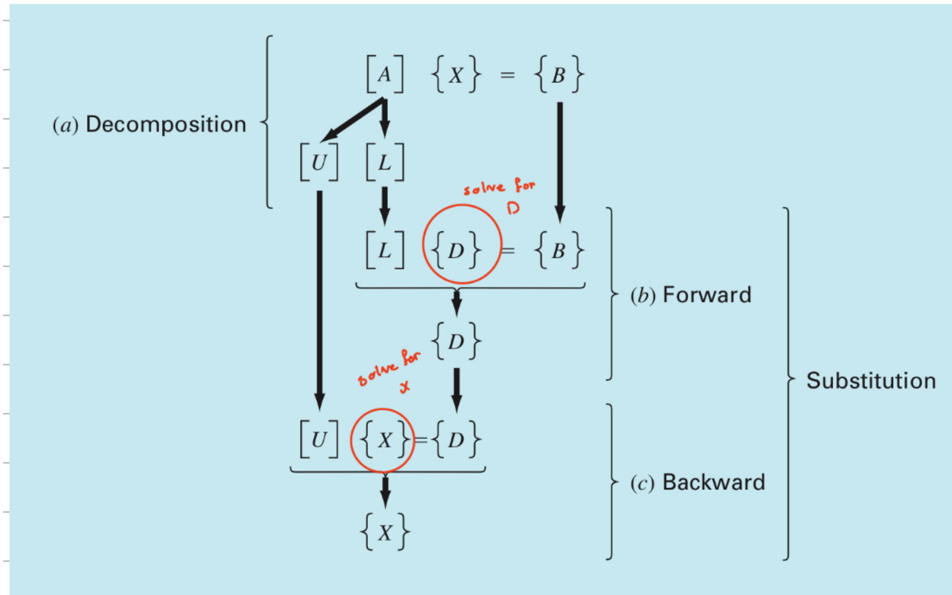
same as what we do in the first step of gauss elimination, to Reduce the system to upper triangular form.

$([U]\{x\} - \{D\} = 0)$

- if we Multiply this equation By a lower Diagonal Matrix ⇒ $[L] = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$

$$[L]\{[U]\{x\} - \{D\}\} = \underbrace{[L][U]\{x\}}_{[A]\{x\}} - \underbrace{[L]\{D\}}_{\{B\}}$$

- [A] ⇒ Decomposed into Upper triangular and lower Diagonal
- {D} ⇒ intermediate vector
- {B} ⇒ used to generate {D} By forward substitution.



↪ we only Do the elimination once !!

↪ in this case, we can Change the (B) without changing the whole solution

$f_{21} = a_{21} / a_{11}$ ↳ in the 1st step

$f_{31} = a_{31} / a_{11}$ ↳ in the 1st step

$f_{32} = a'_{32} / a'_{22}$ ↳ in the 2nd step

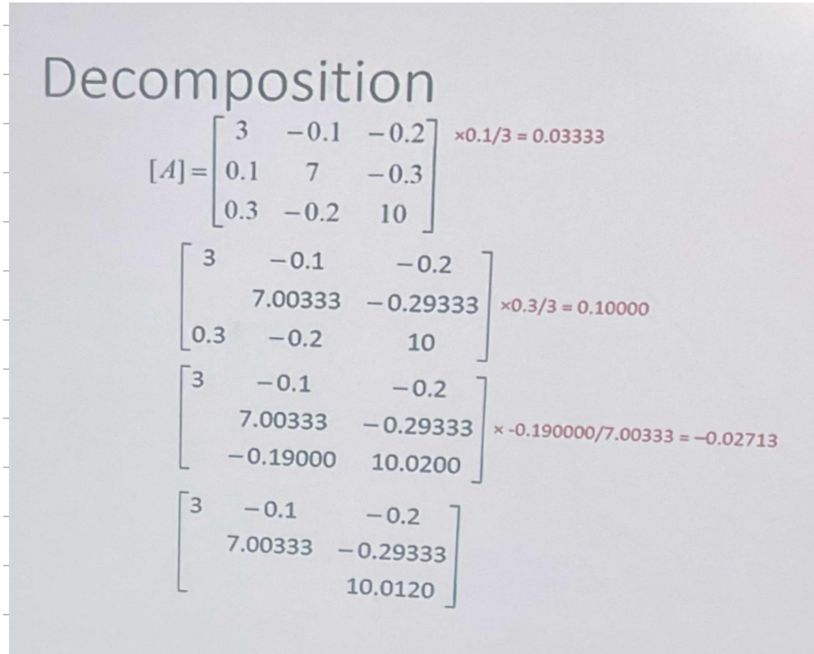
→ Remember :

- 1st step in gauss elimination is that we Multiply the first Row By $f_{21} \Rightarrow \frac{a_{21}}{a_{11}}$
- then we substract the Result from the 2nd Row to eliminate a_{21} and so on

EXAMPLE 10.1 *LU* Decomposition with Gauss Elimination

Problem Statement. Derive an *LU* decomposition based on the Gauss elimination performed in Example 9.5.

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 + 0.2x_2 + 10x_3 = 71.3$$



→ After we did the forward elimination, we get the following Upper Triangular matrix.

$$[U] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.29333 \\ 0 & 0 & 10.0120 \end{bmatrix}$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix}$$

$f_{21} = 0.1/3 = 0.033333$

$f_{31} = 0.3/3 = 0.100000$

$f_{32} = -0.19/7.00333 = -0.0271300$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0.03333 & 1 & 0 \\ 0.10000 & -0.0271300 & 1 \end{bmatrix}$$

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 0.033333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix} \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.29333 \\ 0 & 0 & 10.0120 \end{bmatrix}$$

$$[A] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.099999 & 7 & -0.3 \\ 0.3 & -0.2 & 9.99996 \end{bmatrix}$$

■ Remember:

– Multiplication of Matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$[A][B] = \begin{bmatrix} 1(5)+3(6) & 2(5)+4(6) \\ 1(7)+3(8) & 2(7)+4(8) \end{bmatrix}$$

EXAMPLE 10.2 The Substitution Steps

Problem Statement. Complete the problem initiated in Example 10.1 by generating the final solution with forward and back substitution.

Solution. As stated above, the intent of forward substitution is to impose the elimination manipulations, that we had formerly applied to $[A]$, on the right-hand-side vector $\{B\}$. Recall that the system being solved in Example 9.5 was

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

→ After we Applying the gauss elimination we get:

⇒ Forward-substitution:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.033333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

$\rightarrow d_1 = 7.85$

$\rightarrow d_2 = -19.3 / 0.033333 = -19.5617$

$\rightarrow d_3 = 71.4 / (0.1\, d_1 - 0.0271300) = 70.0843$

⇒ Back substitution:

$[U]\{X\} = \{D\}$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.29333 \\ 0 & 0 & 10.0120 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.5617 \\ 70.0843 \end{Bmatrix}$$

$\rightarrow x_3 = 70.0843 / 10.0120 =$

$x_2 = (-19.5617 - 0.29333\, x_3) / 7.00333 \Rightarrow x_2 = -2.5$

$x_1 = (7.85 + 0.1\, x_2 + 0.2\, x_3) / 3 \Rightarrow x_1 = 3$

10.2: Matrix inverse:

- We compute the inverse of a Matrix, column By Column

• column 1 $\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 =$

• column 2 $\Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow x_2 =$

• column 3 $\Rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow x_3 =$

$\Rightarrow [A]^{-1} = [x_1 \ x_2 \ x_3]$

→ the inverse is defined as:

$[A][A]^{-1} = [I]$ → identity Matrix

→ The inverse can be found from:

$[L]\{d\} = \{b_j\}$ → The column of the inverse Matrix

$[U]\{x_j\} = \{d\}$

→ indications of ill-conditioned systems:

1. if there are any elements in $[A]^{-1} \gg 1$
2. if $[A][A]^{-1}$ is Different from $[I]$
3. if the inverse $[A]^{-1}$ is Different from $[A]$.

EXAMPLE 10.3 Matrix Inversion

Problem Statement. Employ *LU* decomposition to determine the matrix inverse for the system from Example 10.2.

$$[A] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix}$$

Recall that the decomposition resulted in the following lower and upper triangular matrices:

$$[U] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.0120 \end{bmatrix} \qquad [L] = \begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix}$$

$[L]\{d\} = \{b_j\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.03333 & 1 & 0 \\ 0.1000 & -0.0271300 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

→ using Forward elimination:

$d_1 = 1$

$d_2 = -0.03333$

$d_3 = -0.1000(1) + 0.0271300(-0.03333) = -0.1009$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.0033 & -0.29333 \\ 0 & 0 & 10.0120 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.0333 \\ -0.1009 \end{Bmatrix}$$

→ using Back substitution:

$x_3 = -0.1009 / 10.0120 = 0.33249$

$x_2 = (-0.0333 + 0.2933(0.33249))/7.0033 = -0.00518$

$x_1 = (1 + 0.2(0.33249) + 0.1(-0.00518))/3 = -0.01008$

→ we do the same for the 2nd and 3rd column.

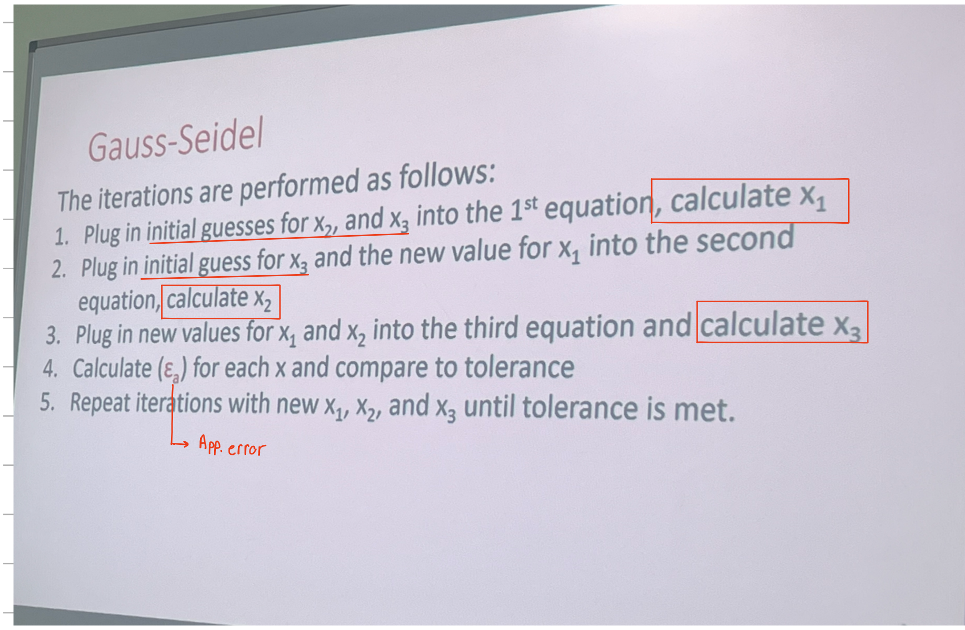
$$\text{so } [A]^{-1} = \begin{bmatrix} 0.33249 & 0.004944 & 0.006798 \\ -0.00518 & 0.142903 & 0.004183 \\ -0.01008 & 0.00271 & 0.09988 \end{bmatrix}$$

$$\begin{matrix} \Downarrow & \Downarrow & \Downarrow \\ \text{from } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \text{from } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \text{from } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{matrix}$$

Chapter (11) :

11.2 : Gauss - Seidel :

- this Method is an alternative for the elimination Methods
- depends on initial guess.



■ **Remember :**

$$\rightarrow \varepsilon_a = \left| \frac{x_i^{j} - x_i^{j-1}}{x_i^{j}} \right| 100\%$$

$\rightarrow \varepsilon_a < \varepsilon_s$ (Tolerance)

$(j) \Rightarrow$ current itteration

$(j-1) \Rightarrow$ previous itteration

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$$
$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3$$

$$x_1 = \frac{b_1 - a_{12} x_2 - a_{13} x_3}{a_{11}} \qquad x_2 = \frac{b_2 - a_{21} x_1 - a_{23} x_3}{a_{22}} \qquad x_3 = \frac{b_3 - a_{31} x_1 - a_{32} x_2}{a_{33}}$$

initial guess

X	0	1	2	- - -
x_1	0			
x_2	0			
x_3	0			
⋮	0			

itiration (1)

EXAMPLE 11.3 Gauss-Seidel Method

Problem Statement. Use the Gauss-Seidel method to obtain the solution of the same system used in Example 10.2:

$$
\begin{aligned}
3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\
0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\
0.3x_1 - 0.2x_2 + 10x_3 &= 71.4
\end{aligned}
$$

Recall that the true solution is $x_1 = 3$, $x_2 = -2.5$, and $x_3 = 7$.

$$x_1 = \frac{7.85 - 0.1x_2 - 0.2x_3}{3}$$

$$x_2 = \frac{-19.3 - 0.1x_1 - 0.3x_3}{7}$$

$$x_3 = \frac{71.4 - 0.3x_1 - 0.2x_2}{10}$$

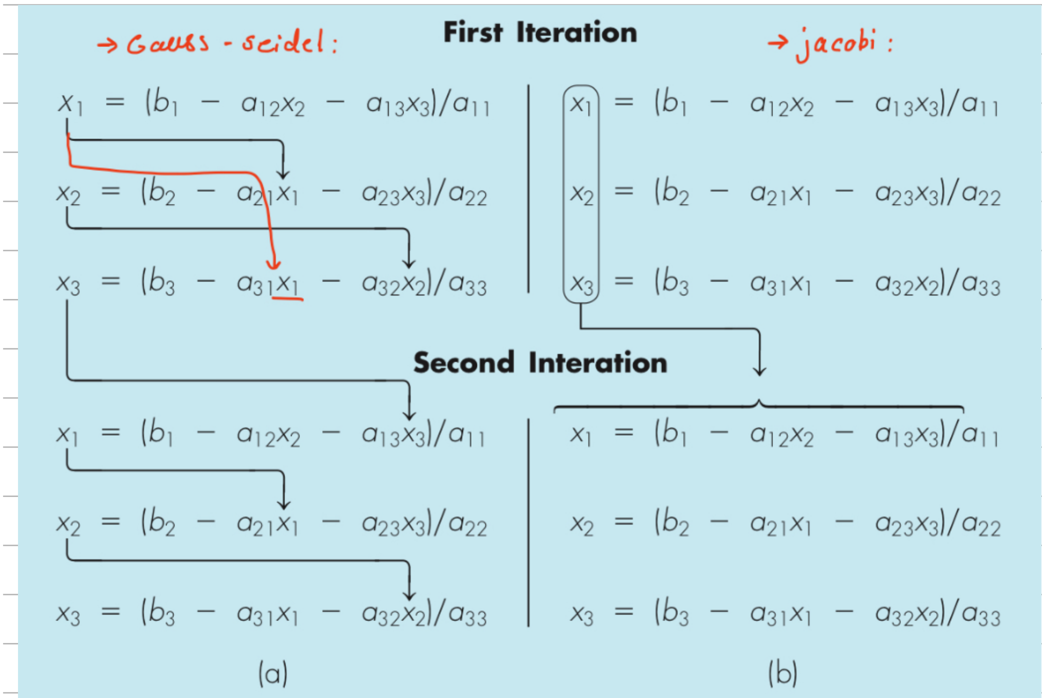
x	0	1	2
x_1	0	2.616	2.9905
x_2	0	-2.794	-2.4996
x_3	0	7.005	7.0003

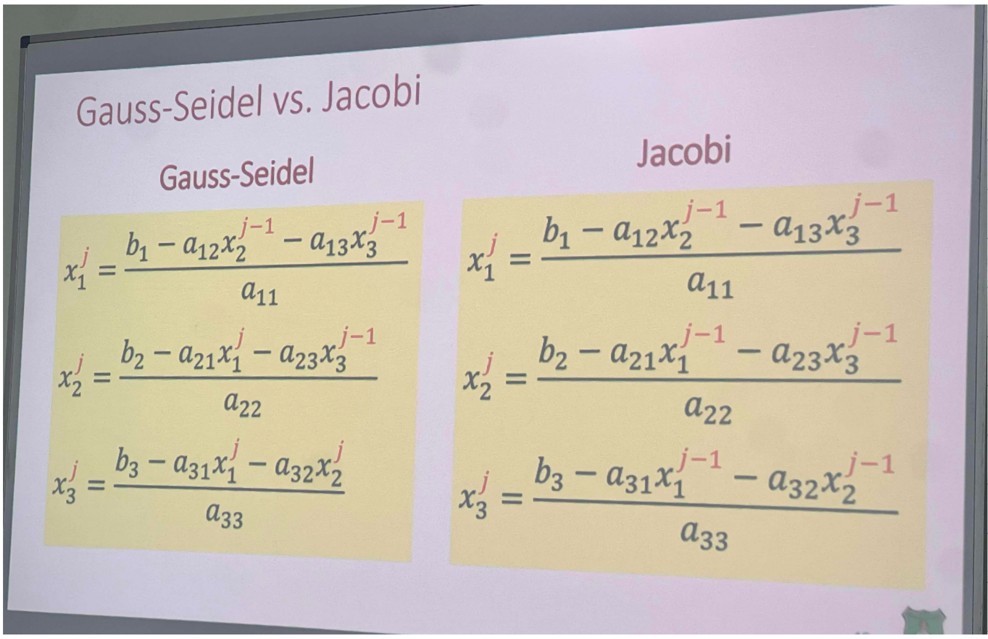
→ we continue the ittirations until we reach an error less than the Toler

$$|\varepsilon_{a,1}| = \left|\frac{2.9905 - 2.616}{2.990557}\right| 100\% = 12.5\%$$

↪ and we do the same for (2) and (3)

→ Difference Between gauss-seidal and jacobi:





Example : (from class)

solve the following equations iteratively :

$6x_1 - 2x_2 + x_3 = 11$

$x_1 + x_2 - 5x_3 = -1$

$-2x_1 + 7x_2 + 2x_3 = 5$

$\Rightarrow$

$6x_1 - 2x_2 + x_3 = 11 \quad 6 > (2+1)$ ✓

$-2x_1 + 7x_2 + 2x_3 = 5 \quad 7 > (2+2)$ ✓

$x_1 + x_2 - 5x_3 = -1 \rightarrow 5 > (1+1)$ ✓

solution ;

$x_1 = \frac{11 + 2x_2 - x_3}{6} =$

$x_2 = \frac{5 + 2x_1 - 2x_3}{7} =$

$x_3 = \frac{-1 - x_2 - x_1}{5} =$

x	0	1	2
x_1	0	1.833	2.0690
x_2	0	1.238	1.305
x_3	0	0 ?	

↪ using jacobi Method :

x	0	1	2
x_1	0	1.833	⋮
x_2	0	0.7143	⋮
x_3	0	-0.7095	⋮

→ when to stop ?

- when the App. error is less than the tolerance.

FIGURE 11.5
Iteration cobwebs illustrating (*a*) convergence and (*b*) divergence of the Gauss-Seidel method. Notice that the same functions are plotted in both cases (u: $11x_1 + 13x_2 = 286$; v: $11x_1 - 9x_2 = 99$). Thus, the order in which the equations are implemented (as depicted by the direction of the first arrow from the origin) dictates whether the computation converges.