Lecture 1: Introduction

→ Creative process

Develop a clear description → Identify the important factors → Propose or refine a model → Manipulate the model → Confirm the sol. → Conc. & recommendation

Conduct experiments

Statistics

↓ Descriptive: Collect / organize / present

↓ Inferential: Analyze / Interpret

*Variability: Successive observations of a system or phenomenon does not produce exactly the same result.

Statistics → gives a framework for describing this variability & for learning about potenial sources of variability.

* The dot diagram is a very useful plot for displaying a small body of data (around 20)
  → allows to see easily two features of the data: location or the middle, scatter or variability

⇒ Basic Methods of collecting Data:

1. Retrospective study using historical data [Data collected in the past for other purposes]
2. Observational study, Data presently collected by a passive observer
3. Designed expirement, Data collected in response to process input changes

Mechanistic & Empirical Methods

a mechanistic model is built from our underlying knowledge of the basic physical mechanisims that relates several variables.

An Empirical Model is built from our engineering & scientific knowledge of the phenomenon, but is not directly developed from our theoretical or first--principles understanding of the underlying mechanisim.

* models can also reflect uncertainty → probability models help quantify the risks involved in statistical interference.

Probability

Probability refers to the study of randomness & uncertainty

→ Experiment: Any process or action that generates a unique observation out from the set of possible observations.

Lecture 2: Concepts in probability

Probability

- The study of Randomness & uncertainity
  - quantify likelihood or chance
  - represent risk or uncertainty
  - Degree of belief or relative frequancy

Random Experiments

An experiment that can result in different outcomes, even though it's repeated in the same mannar.

Sample Space

The set of all possible unique outcomes of a random experiment (S)

Ex: Coin toss $S = \{ \text{heads}, \text{tails} \}$

Roll of dice $S = \{1, 2, 3, 4, 5, 6\}$

Consider an experiment in which you select a molded plastic part, such as a connector, and measure its thickness. The possible values for thickness depend on the resolution of the measuring instrument, and they also depend on upper and lower bounds for thickness. However, it might be convenient to define the sample space as simply the positive real line

$$S = R^+ = \{ x > 0 \}$$

because a negative value for thickness cannot occur.

If it is known that all connectors will be between 10 and 11 millimeters thick, the sample space could be

$$S = \{ 10 < x < 11 \}$$

If the objective of the analysis is to consider only whether a particular part is low, medium, or high for thickness, the sample space might be taken to be the set of three outcomes:

$$S = \{ \text{low}, \text{medium}, \text{high} \}$$

* qualitative outcomes

If the objective of the analysis is to consider only whether or not a particular part conforms to the manufacturing specifications, the sample space might be simplified to the set of two outcomes

$S = \{yes, no\}$

Example 2-2 If two connectors are selected and measured, the extension of the positive real line R is to take the sample space to be the positive quadrant of the plane:

$$S = R^{+} \times R^{+}$$ ← two connectors

If the objective of the analysis is to consider only whether or not the parts conform to the manufacturing specifications, either part may or may not conform. We abbreviate *yes* and *no* as *y* and *n*. If the ordered pair *yn* indicates that the first connector conforms and the second does not, the sample space can be represented by the four outcomes:

$S = \{yy, yn, nn, ny\}$

If we are only interested in the number of conforming parts in the sample, we might summarize the sample space as

$S = \{0, 1, 2\}$ * no conforming, 1 conforming, 2 conforming

As another example, consider an experiment in which the thickness is measured until a connector fails to meet the specifications. The sample space can be represented as

$S = \{n, yn, yyn, yyyn, yyyyn, \ldots\}$ * countably infinite

Types of Sample Spaces

Sample space
- Discrete → finite
  → countably infinite
  ↓ nothing between points
- Continous [Uncountably infinite]
  ↳ like between 10 & 11
  10.1 10.2 10.3 ...

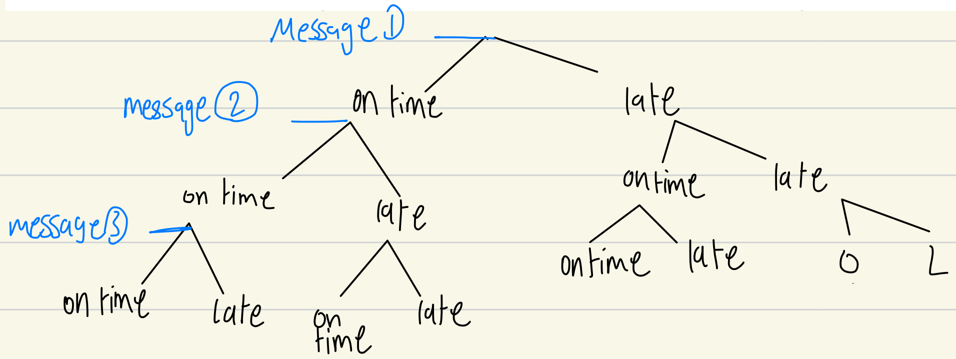
Tree Diagrams

When a sample space can be constructed in several steps or stages, we can represent each of the n_1 ways of completing the first step as a branch of a tree

Completing the second step n_2 [branches starting from the ends of the original branches]

Each message in a digital communication system is classified as to whether it is received within the time specified by the system design. If three messages are classified, use a tree diagram to represent the sample space of possible outcomes.

Each message can either be received on time or late. The possible results for three messages can be displayed by eight branches in the tree diagram shown in Fig. 2-5.



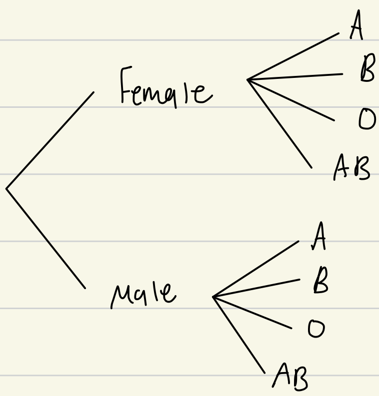
$S = \{ OOO, OOL, OLO, OLL, LOO, LOL, LLO, LLL \}$

\# of messages.

$\star$ $2^3 = 8$

options

Ex: Male & Female Blood types



$S = \{ FA, FB, FO, FAB, MA, MB, MO, MAB \}$

* Tree diagrams are not necessarly from the same genre.

Events:

Any subset of the sample space of a random experiment

Example 2-6

Consider the sample space $S = \{yy, yn, ny, nn\}$ in Example 2-2. Suppose that the set of all outcomes for which at least one part conforms is denoted as E_1. Then,

$$E_1 = \{yy, yn, ny\}$$

The event in which both parts do not conform, denoted as E_2, contains only the single outcome, $E_2 = \{nn\}$. Other examples of events are $E_3 = \emptyset$, the null set, and $E_4 = S$, the sample space. If $E_5 = \{yn, ny, nn\}$,

$E_1 \cup E_5 = S$ ↳ Union

$E_1 \cap E_2 = \{yn, ny\}$ ↳ Intersection

$E_1' = \{nn\}$ ↳ Compliment

Types of Events:

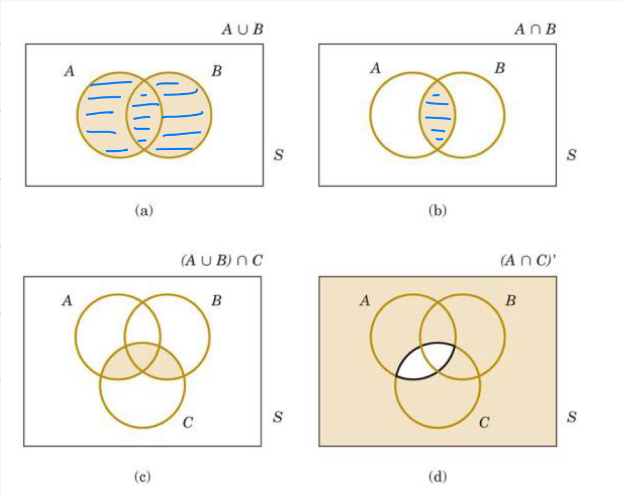
1. Empty set $\emptyset$: Impossible event
2. The subset S : certain event
3. Simple / elementary : one outcome only
4. Compound : more than one outcome

Basic Set operations – Venn Diagrams

Union $A \cup B$ All outcomes that are contained in either two events

Intersection $A \cap B$: All outcomes contained in both [Joint probability]

Compliment A' : outcomes in S that are not in A



Measurements of the thickness of a part are modeled with the sample space: $S = R^{+}$.

Let $E_1 = \{x \mid 10 \le x < 12\}$,

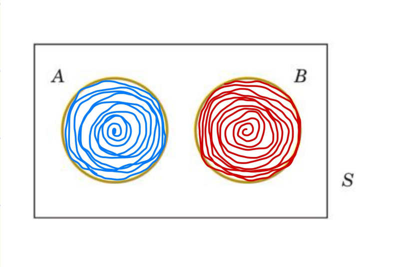
Let $E_2 = \{x \mid 11 < x < 15\}$

$E_1 \cup E_2 = \{ 10 \le x < 15 \}$

$E_1 \cap E_2 = \{ 11 < x < 12 \}$

$E_1' = \{ 0 < x < 10 , x \geq 12 \}$

$E_1' \cap E_2 = \{ 12 \le x < 15 \}$



Mutually Exclusive [Disjoint] Events

→ $E_1 \cap E_2 = \emptyset$ ⇒ mutually exclusive

→ The occurence of one event precludes the occurance of the other Ex: { success , failure } , { alive , dead }

Mutually Exclusive Events – Laws

Commutative Law

$A \cup B = B \cup A$

$A \cap B = B \cap A$

Distributive Law

$(A \cup B) \cap C = (A \cup C) \cap (B \cup C)$

$(A \cap B) \cup C = (A \cap C) \cup (B \cap C)$

Associative Law

$(A \cap B) \cap C = A \cap (B \cap C)$

$(A \cup B) \cup C = A \cup (B \cup C)$

De-Morgan's Laws

$(A \cup B)' = A' \cap B'$ ⇒ The compliment of the union is the intersection of the compliments

$(A \cap B)' = A' \cup B'$ ⇒ The compliment of the intersection is the union of the compliments.

$(A')' = A$

Axioms of probability

1. The probability of an event is a real number greater than or equal to zero

$$0 \leq P(E) \leq 1$$

2. The probability that atleast one of all the possible outcomes of a process will occur is 1

$$P(S) = 1 \qquad P(\phi) = 0$$

$$P(E') = 1 - P(E)$$

3. if E_1 & E_2 are mutually Exclusive $\quad P(E_1 \cup E_2) = P(E_1) + P(E_2)$

Types of probability

→ Subject [Degree of belief]

→ Relative frequancy

Probability based on Equally-likely outcomes.

whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $1/N$

Probability of an Event

for a discrete sample space, the probability of an event E, denoted by $P(E)$ = sum of the probabilities of the outcomes in E

Probability of joint Events

Joint Events are generated by applying basic set operations to indivisual Events

	B	B'	
A	$P(A \cap B)$	$P(A \cap B')$	$P(A)$
A'	$P(A' \cap B)$	$P(A' \cap B')$	$P(A')$
	$P(B)$	$P(B')$	$P(S) = 1$

Addition Rules [Probability of a union]

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$P(A \cup B) = P(A) + P(B)$ * mutually Exclusive $A \cap B = \emptyset$

De Morgan's Law

$(A \cup B)' = A' \cap B'$ $\qquad$ $(A \cap B)' = A' \cup B'$

⇒ 3 or more Events

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$P(E_1 \cup E_2 \cup E_3 \cdots) = \sum P(E_j)$ * Mutually Exclusive $E_i \cap E_j = \emptyset$

Lecture 3: Concepts in Probability 2

Determine the number of outcomes in Events
- → Multiplication Rule
- → Permutation Rule
- → Combination Rule

Multiplication Rule

Total number of Arrangements (TNA) $= \prod_{i=1}^{k} n_i$

Permutation Rule

A unique sequance of distinct items [order matters]

Used to find the possible number of arrangements when there is only one group of objects

Arrangement of r objects selected from a single group of n possible objects.

ways the objects can be arranged ← $P_r^n = \dfrac{n!}{(n-r)!}$

Ex: 13 objects 2 subsets

$$P_r^{13} = \frac{13!}{(13-2)!} = \frac{13!}{11!} = \frac{13 \times 12 \times \cancel{11!}}{\cancel{11!}} = 156$$

* A Permutation is an ordered arrangement of all or some of the elements of a set.

* The order of the objects of each possible outcome is different.

→ Similar item Permutations

Counting the sequence when some items are identical $\dfrac{n!}{n_1!\, n_2! \cdots}$

Combination Rule

A combination is a selection of r items from a set of n where order does not matter

Number of permutations ≥ number of combinations

Since order doesn't matter with combinations, we divide # of permutations by r!, where r! is the # of arrangements of r elements

$$C_r^n = \frac{n!}{r!(n-r)!}$$

Lecture 4: Conditional Probability

probability of a particular event occuring given that another event has occurred.

$P(B|A)$

probability of B given that A

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$P(A)$ = number of outcomes A /n

$P(A \cap B)$ = number of outcomes in $A \cap B$ /n

$P(A|B) \neq P(B|A)$

Parts Classified			
Defective	Surface Flaws		Total
	Yes (F)	No (F')	
Yes (D)	10	18	28
No (D')	30	342	372
Total	40	360	400

$P(F) = \frac{40}{400}$ $\qquad P(D) = \frac{28}{400}$

$$P(D|F) = \frac{P(F \cap D)}{P(F)} = \frac{\frac{10}{400}}{40/400}$$

$$P(F|D) = \frac{P(D \cap F)}{P(D)} = \frac{10/400}{28/400}$$

$$P(D'|F) = \frac{P(F \cap D')}{P(F)} = \frac{30/400}{40/400}$$

$$P(D|F') = \frac{P(F' \cap D)}{P(F')} = \frac{18/400}{360/400}$$

$$P(D'|F') = \frac{P(D' \cap F')}{P(F')} = \frac{342/400}{360/400}$$

Random Samples:

Random = Equally likely

ordered → Permutation

unordered → Combination

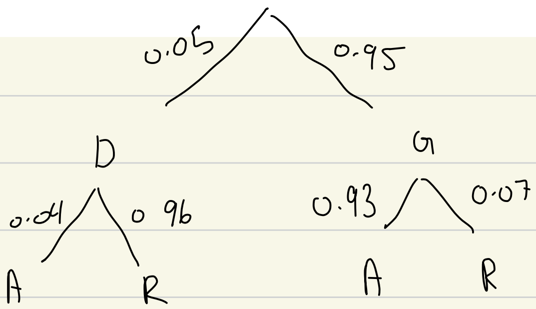
Multiplication Rule:

$$P(A \cap B) = P(B)\,P(A|B) = P(A)\,P(B|A)$$

for statistically independent $P(A \cap A_2 \cap A_3 \cdots) = \prod P(A_i)$

A company produces machine components which pass through an automatic testing machine. 5% of the components entering the testing machine are defective. However, the machine is not entirely reliable. If a component is defective there is 4% probability that it will not be rejected. If a component is not defective there is 7% probability that it will be rejected.

a) What fraction of all the components are rejected?
b) What fraction of the components rejected are actually not defective?
c) What fraction of those not rejected are defective?



Rejected $= 0.05(0.96) + 0.95(0.07)$

$= 0.1145$

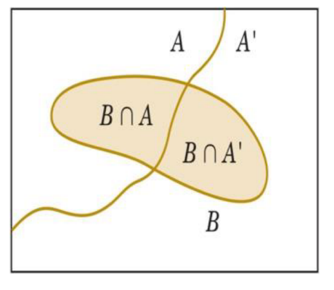
Accepted $= 0.05(0.04) + 0.95(0.93)$

$= 0.8855$

Rejected but good ⇒ $P(G|R) = \dfrac{P(G \cap R)}{P(R)} = \dfrac{0.95(0.07)}{0.1145} = 0.58$

Not Rejected are Defective ⇒ $P(D|A) = \dfrac{P(D \cap A)}{P(A)} = \dfrac{0.05(0.04)}{0.8855} = 0.0026$

Total Probability Rule



A & A' are mutually exclusive

$P(B) = P(B \cap A) + P(B \cap A')$

$P(B|A)P(A) + P(B|A')P(A')$

Example 2-14: Semiconductor Contamination

Information about product failure based on chip manufacturing process contamination is given below. Find the probability of failure.

Probability of Failure	Level of Contamination	Probability of Level
0.100	High	0.2
0.005	Not High	0.8

$P(fail \mid h) = 0.1 \qquad P(h) = 0.2$

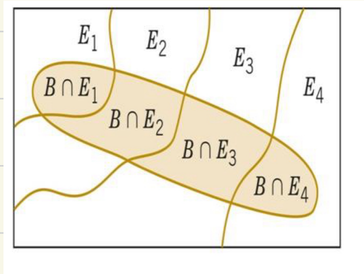
$P(fail \cap h) = 0.1(0.2) = 0.02$

$P(fail \mid nh) = 0.005 \qquad P(nh) = 0.8$

$P(fail \cap nh) = 0.8(0.005) = 0.004$

$P(F) = P(Fail \cap h) + P(fail \cap nh) = 0.02 + 0.004 = 0.024$

Total Probability Rule [Multiple Events]



E_1 & E_2 & E_3 ... E_k are mutually exclusive & exhaustive

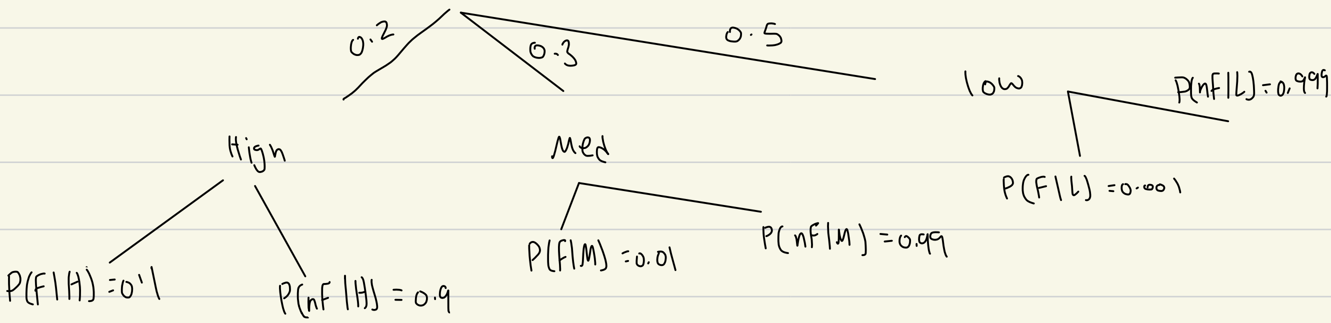
$$P(B) = P(B \cap E_1) + P(B \cap E_2) + P(B \cap E_3) \cdots P(B \cap E_k)$$

$$P(B \mid E_1) P(E_1) + P(B \mid E_2) P(E_2) + P(B \mid E_3) P(E_3) \cdots P(B \mid E_k) P(E_k)$$

Example 2-15: Semiconductor Failures-1

Continuing the discussion of contamination during chip manufacture, find the probability of failure.

Probability of Failure	Level of Contamination	Probability of Level
0.100	High	0.2
0.010	Medium	0.3
0.001	Low	0.5



$$P(F) = 0.2(0.1) + 0.3(0.01) + 0.5(0.001) = 0.0235$$

Event Independence

Two events are independent if:

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

$$P(A \cap B) = P(A)\,P(B)$$

→ The occurrence of one event has no impact on the probability of occurrence of the other event.

$$P(E_1 \cap E_2 \cap E_3 \cdots \cap E_k) = P(E_1)\,P(E_2)\,P(E_3) \cdots P(E_k)$$

Example 2-16: Flaws and Functions

Table 1 provides an example of 400 parts classified by surface flaws and as (functionally) defective. Suppose that the situation is different and follows Table 2. Let F denote the event that the part has surface flaws. Let D denote the event that the part is defective.

The data shows whether the events are independent.

TABLE 1 Parts Classified			
	Surface Flaws		
Defective	Yes (F)	No (F')	Total
Yes (D)	10	18	28
No (D')	30	342	372
Total	40	360	400

TABLE 2 Parts Classified (data chg'd)			
	Surface Flaws		
Defective	Yes (F)	No (F')	Total
Yes (D)	2	18	20
No (D')	38	342	380
Total	40	360	400

$$P(D|F) = \frac{10/400}{40/400} = 0.25$$

$$P(D) = \frac{28}{400} = 0.07$$

* F & D ⇒ Dependent

$$P(D|F) = \frac{2}{400} \div \frac{40}{400} = 0.05$$

$$P(D) = \frac{20}{400} = 0.05$$

* F & D ⇒ Independent

Example 2-17: Semiconductor Wafers

Assume the probability that a wafer contains a large particle of contamination is 0.01 and that the wafers are independent; that is, the probability that a wafer contains a large particle does not depend on the characteristics of any of the other wafers. If 15 wafers are analyzed, what is the probability that no large particles are found?

$P(E_i) = 0.99$ → i^{th} wafer with no large particles

$$P(E_1 \cap E_2 \cdots E_i) = (0.99)^{15} = 0.86$$

Bayes Theorem

Example 2-18

The conditional probability that a high level of contamination was present when a failure occurred is to be determined. The information from Example 2-14 is summarized here.

find $P(F)$

Probability of Failure	Level of Contamination	Probability of Level
0.100	High	0.2
0.005	Not High	0.8

Solution:

$P(F|H) = 0.1$ $\quad P(H) = 0.2$

$P(F|nH) = 0.005$ $\quad P(nH) = 0.8$

$$P(H|F) = \frac{P(F|H)\,P(H)}{P(F)}$$

$$P(H|F) = \frac{0.1\,(0.2)}{0.024} = 0.83$$

$$P(F) = 0.2(0.1) + 0.8(0.005) = 0.024$$

Example 2-19: Bayesian Network

A printer manufacturer obtained the following three types of printer failure probabilities. Hardware $P(H) = 0.3$, software $P(S) = 0.6$, and other $P(O) = 0.1$. Also, $P(F \mid H) = 0.9$, $P(F \mid S) = 0.2$, and $P(F \mid O) = 0.5$. If a failure occurs, determine if it's most likely due to hardware, software, or other.

$P(H) = 0.3$

$P(S) = 0.6$

$P(O) = 0.1$

$P(F|H) = 0.9$

$P(F|S) = 0.2$

$P(F|O) = 0.5$

$$P(F) = 0.3(0.9) + 0.6(0.2) + 0.1(0.5) = 0.44$$

$$P(H|F) = \frac{P(F|H) \cdot P(H)}{P(F)} = \frac{0.9(0.3)}{0.44} = 0.613$$

$$P(S|F) = \frac{P(F|S)\;P(S)}{P(F)} = \frac{0.2(0.6)}{0.44} = 0.27$$

$$P(O|F) = \frac{P(F|O)\,P(O)}{P(F)} = \frac{0.5(0.1)}{0.44} = 0.11$$

Example on Bayes' Theorem

Known: Vendors I, II, III, and IV provide all the bushings to a certain factory. The following table shows the amounts supplied and percentage of good parts supplied by the vendors.
Wanted: What is the probability that a randomly selected bushing is bad? What is the probability it came from vendor 3?

Vendor	% Supply	% Good	% Bad
I	25	80	20
II	35	95	5
III	10	70	30
IV	30	90	10

$$P(B) = 0.25(0.2) + 0.35(0.05) + 0.1(0.3) + 0.3(0.1) = 0.1275$$

$$P(V_3 \mid B) = \frac{P(V_3 \cap B)}{P(B)} = \frac{0.1(0.3)}{0.1275} = 0.2353$$

Lecture 5 : Discrete Probability Distributions

→ Random variable.

function that assigns a real number to each outcome in the sample space of random experiment

[X]

→ A Discrete Random variable

Random variable with finite (or countable Infinite) range.

Probability mass function

$f(x_i) \geq 0$ ⇝ non negative probabilities

$\Sigma f(x_i) = 1$ ⇝ Sum of probabilities = 1

$f(x_i) = P(X = x_i)$

x_i	x_1	x_2	x_3
$f(x_i) = P(x_i)$	$P(x_1)$	$P(x_2)$	$P(x_3)$

Ex:

x	-2	-1	0	1	2
$P(x)$	1/8	2/8	2/8	2/8	1/8

verify the function is a probability mass function $\Sigma P(x) = 1$

find $P(X \leq 2)$ $P(2) + P(1) + P(0) + P(-1) + P(-2) = 1$

find $P(X > -2)$ $P(-1) + P(0) + P(1) + P(2) = 7/8$

$P(-1 \leq X \leq 1)$ $P(-1) + P(0) + P(1) = 6/8$

$P(X \leq -1 \text{ or } X = 2)$ $P(-1) + P(-2) + P(2) = 4/8$

↳ Union $P(A) + P(B) - P(A \cap B)$

Ex: Shipment of 20 Laptops (3 defective), Random Purchase of 2, find the probability distribution for the number of Defectives

→ Possible values 0, 1, 2

x	0	1	2
$P(x)$	$68/95$	$51/190$	$3/190$

→ Probability mass function

$$P(0) = \frac{\binom{3}{0}\binom{17}{2}}{\binom{20}{2}} = \frac{68}{95}$$

$$P(1) = \frac{\binom{3}{1}\binom{17}{1}}{\binom{20}{2}} = 51/190$$

$$P(2) = \frac{\binom{3}{2}\binom{17}{0}}{\binom{20}{2}} = 3/190$$

Cumulative Distribution

$$F(x) = P(X \leq x) = \sum f(x_i)$$

Ex:

	x	-2	-1	0	1	2
	$P(x)$	$1/8$	$2/8$	$2/8$	$2/8$	$1/8$
CDF →	$F(x)$	$1/8$	$3/8$	$5/8$	$7/8$	$8/8$

Ex:

x	0	1	2	3	4
$P(x)$	0.6561	0.2916	0.0486	0.0036	0.0001
$F(x)$	0.6561	0.9477	0.9963	0.9999	1

Mean & Variance

→ Mean is a measure of the center or the middle of the probability distribution

→ Variance is a measure of the dispersion, or variability in the distribution

Mean or Expected value $\mu = E(x) = \sum x f(x)$

Variance $\sigma^2 = V(x) = \sum x^2 f(x) - \mu^2$

Standard deviation $\sigma = \sqrt{\sigma^2}$

Ex:

x	-2	-1	0	1	2
$P(x)$	$1/8$	$2/8$	$2/8$	$2/8$	$1/8$

Mean: $\mu = -2(1/8) + -1(2/8) + 0(2/8) + 1(2/8) + 2(1/8) = 0$

Variance $\sigma^2 = \left[-2^2(1/8) + -1^2(2/8) + 0^2(2/8) + 1^2(2/8) + 2^2(1/8)\right] - 0^2$

$\sigma^2 = 1.5$

Standard Deviation $\sigma = \sqrt{1.5} = \sqrt{6}/2$

Ex: lot contianing 7 components → 4 good, 3 defective, A sample of 3 is taken by inspector

Find the expected value of good components

X → # of good components from 4

x	0	1	2	3
$P(x)$	$1/35$	$12/35$	$18/35$	$4/35$

$P(0) = \dfrac{\binom{3}{3}\binom{4}{0}}{\binom{7}{3}} = 1/35$

$P(2) = \dfrac{\binom{3}{1}\binom{4}{2}}{\binom{7}{3}} = 18/35$

$P(1) = \dfrac{\binom{3}{2}\binom{4}{1}}{\binom{7}{3}} = 12/35$

$P(3) = \dfrac{\binom{3}{0}\binom{4}{3}}{\binom{7}{3}} = 4/35$

$$E(x) = 0\left(\tfrac{1}{35}\right) + 1\left(\tfrac{12}{35}\right) + 2\left(\tfrac{18}{35}\right) + 3\left(\tfrac{4}{35}\right) = \tfrac{12}{7} \doteq 1.7$$

$$V(x) = \tfrac{24}{7} - \left(\tfrac{12}{7}\right)^2 = \tfrac{24}{49}$$

$$\sigma = \sqrt{\tfrac{24}{49}} \doteq 0.7$$

→ For any constants a & b

$$E(a) = a$$

$$E(ax+b) = aE(x) + b$$

$$V(a) = 0$$

$$V(ax+b) = a^2 V(x)$$

Ex: X is a random variable with mean = 6 & variance = 100

$Y = 3X + 6$ find mean & variance for Y

$$E(Y) = 3E(X) + 6 = 3(6) + 6 = 24$$

$$V(Y) = 3^2 V(X) = 3^2(100) = 900$$

Uniform Probability Distribution

→ A random variable X has a discrete uniform distribution if each of the n values in it's range have equal probability

$$f(x) = \frac{1}{n}$$

$$\mu(X) = \frac{a+b}{2}$$

$$\sigma^2 = \frac{(b-a+1)^2 - 1}{12}$$

Let the random variable X denote the number of the 48 voice lines that are in use at a particular time. Assume that X is a discrete uniform random variable with a range of 0 to 48. Find $E(X)$ & σ

$$E(x) = \frac{b+a}{2} = \frac{48+0}{2} = 24$$

$$\sigma^2 = \frac{(b-a+1)^2 - 1}{12} = \frac{(48-0+1)^2 - 1}{12} = 200$$

$$\sigma = \sqrt{200} = 14.14$$

Bernoulli Trials & Distribution

→ A trial with only two possible mutually exclusive outcomes (success, failure)

n Bernoulli trials such that:

- Trials are statistically independent
- Each trail has two possible outcomes (success, fail)
- The probability of success in each trial remains constant

Success p	$X = 1$	$p + q = 1$
failure q	$X = 0$	

* The random variable X that equals the number of trials that result in a success has a binomial random variable with $0 < p < 1$

$n = 1, 2, \ldots$

Mean $\mu = E(x) = np$

Variance $\sigma^2 = V(x) = np(1-p)$

S.D $\sigma = \sqrt{\sigma}$

→ Probability mass function $f(x) = \binom{n}{x} p^x (1-p)^{n-x}$

Ex: Rolling a die 4 times, Probability of rolling a 1 twice

$n=4$ $\quad f(x) = \binom{4}{2}\left(\frac{1}{6}\right)^2\left(1-\frac{1}{6}\right)^{4-2} = \frac{25}{216} = 0.1157$

$x=2$

$p=\frac{1}{6}$ $\quad E(x) = np = 4\left(\frac{1}{6}\right) = 0.666$

$V(x) = np(1-p) = 4\left(\frac{1}{6}\right)\left(1-\frac{1}{6}\right) = 0.555$

$\sigma = 0.745$

Each sample of water has a 10% chance of containing a particular organic pollutant. Assume that the samples are independent with regard to the presence of the pollutant. Find the probability that, in the next 18 samples, exactly 2 contain the pollutant.

$n=18$

$x=2$ $\quad f(x) = \binom{n}{x} p^x (1-p)^{n-x}$

$p=0.10$

$\binom{18}{2}(0.1)^2(1-0.1)^{18-2} = 0.2835$

Determine the probability that at least 4 samples contain the pollutant.

$P(X \geq 4) = 1 - P(X<4)$

$1 - \sum_{x=0}^{3}\binom{18}{x}(0.1)^x(1-0.1)^{18-x}$

$1 - [0.15 + 0.3 + 0.284 + 0.168] = 0.098$

Now determine the probability that $3 \leq X < 7$.

$P(3 \leq X < 7) = \sum_{x=3}^{6}\binom{18}{x}(0.1)^x(1-0.1)^{18-x}$

$= 0.168 + 0.07 + 0.0217 + 0.00524 = 0.265$

For the number of transmitted bit received in error in Example 3-16, $n = 4$ and $p = 0.1$. Find the mean and variance of the binomial random variable.

$n=4$

$p=0.1$

$$\text{mean} = \mu = E(X) = np$$

$$4(0.1) = 0.4$$

$$\text{Variance} = \sigma^2 = V(X) = np(1-p)$$

$$4(0.1)(1-0.1) = 0.36$$

$$\sigma = \sqrt{0.36} = 0.6$$

Geometric Distribution

Binomial Distribution ↓ Probability of getting X success

Geometric Distribution ↓ Probability of getting first success on trial # X

→ Mean $\mu = E(X) = \frac{1}{p}$

→ Variance $\sigma^2 = V(X) = \frac{(1-p)}{p^2}$

→ Probability $f(x) = p(1-p)^{x-1}$

Ex: odds of making it through the light = 0.2, how many lights you expect to hit before making it through one? SD? probability of 3rd light being the first green?

$p = 0.2$

$x = 3$

1. $\mu = \frac{1}{p} = \frac{1}{0.2} = 5$ lights.

2. $\sigma^2 = \frac{(1-p)}{p^2} = \frac{(1-0.2)}{0.2^2} = 20$

$SD = \sqrt{20} = 4.47$ lights

3. $f(x) = p(1-p)^{x-1}$

$0.2(1-0.2)^{3-1} = 0.128$

Example 3.21: Wafer Contamination

The probability that a wafer contains a large particle of contamination is 0.01. Assume that the wafers are independent. What is the probability that exactly 125 wafers need to be analyzed before a particle is detected?

$p = 0.01$

$x = 125$

$f(x) = p(1-p)^{x-1}$

$0.01(1-0.01)^{125-1} = 2.87 \times 10^{-3}$

Example (Redundancy)

Suppose that a primary device has failed as a result of a high temperature environment. The probability that an electronic switch will successfully activate a backup device is 0.6. If switch failures are statistically independent and the switches are tried one at a time, how many parallel switches are required to achieve at least 95% probability of successful switching?

$p = 0.6$

→ countably infinite sample

X	1	2	3	4	...
F(X)	0.6	0.84	0.936	0.9744	

$f(x) = p(1-p)^{x-1}$

$0.6(1-0.6)^{2-1} = 0.24 + 0.6 = 0.84$

$0.6(1-0.6)^{3-1} = 0.096 + 0.84 = 0.936$

$0.6(1-0.6)^{4-1} = 0.0384 + 0.936 = 0.9744$

to get at least 95% → 4 switches

A chemical process plant has a pump that will operate with a probability of 0.9. How many pumps, in parallel, do you need to have a probability of the pump(s) operating greater than 0.99?

$p = 0.9$

X	1	2	3
$F(X)$	0.9	0.99	0.999

$0.9(1-0.9)^{1-1} = 0.9$

$0.9(1-0.9)^{2-1} = 0.09 + 0.9 = 0.99$

$0.9(1-0.9)^{3-1} = 0.009 + 0.99 = 0.999$

Probability greater than 99% $\Rightarrow$ 3 pumps

Negative Binomial Distribution.

X denotes the number of trials till the first success

$X \rightarrow$ negative Binomial random variable with $0 < p < 1$ $r = 1, 2, \cdots$

$\rightarrow$ The geometric distribution is a special case of negative binomial when $r = 1$

Mean $\mu = E(x) = \frac{r}{p}$

Variance $\sigma^2 = V(x) = \frac{r(1-p)}{p^2}$

probability $f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$ $x = r, r+1, r+2, \cdots$

Binomial	Negative Binomial
$\downarrow$	$\downarrow$
$X \rightarrow$ number of successes	$X \rightarrow$ number of trials
$n \rightarrow$ fixed number of trials	$n \rightarrow$ fixed number of successes

$p = 0.09$

$r = 3$

$x = 10$

$$f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

$$\binom{10-1}{3-1} 0.09^3 (1-0.09)^{10-3} = 0.0135$$

EXAMPLE 3.22 Camera Flashes
Consider the time to recharge the flash in Example 3.1. The probability that a camera passes the test is 0.8, and the cameras perform independently. What is the probability that the third failure is obtained in five or fewer tests?

$p = (1-0.8) = 0.2$

$x \leq 5$

$r = 3$

$$f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

$$P(x \leq 5) = \sum_{x=3}^{5} \binom{x-1}{r-1} (0.2)^r (1-0.2)^{x-r}$$

$x=3$ $\quad \binom{2}{2} (0.2)^3 (1-0.2)^{3-3} = 0.008$

$x=4$ $\quad \binom{3}{2} (0.2)^3 (1-0.2)^{4-3} = 0.0192$

$x=5$ $\quad \binom{4}{2} (0.2)^3 (1-0.2)^{5-3} = 0.03072$

$\Sigma = 0.05792$

Hyper Geometric Distribution

→ finite population, without replacement

→ Typically the sample size is greater than 5% of the size of population

$$f(x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \qquad x = \max\{0, n+k-N\}, \min\{k, n\}$$

Mean $\mu = E(x) = np$

Variance $\sigma^2 = V(x) = np(1-p)\frac{N-n}{N-1}$

Example 3-27: Parts from Suppliers-1

A batch of parts contains 100 parts from supplier A and 200 parts from Supplier B. If 4 parts are selected randomly, without replacement, what is the probability that they are all from Supplier A?

$$\frac{\binom{100}{4}\binom{200}{0}}{\binom{300}{4}} = 0.0118$$

What is the probability that two or more parts are from supplier A?

$$P(x \geq 2)$$

$$\frac{\binom{100}{2}\binom{200}{2}}{\binom{300}{4}} + \frac{\binom{100}{3}\binom{200}{1}}{\binom{300}{4}} + \frac{\binom{100}{4}\binom{200}{0}}{\binom{300}{4}} = 0.407$$

What is the probability that at least one part in the sample is from Supplier A?

$$P(X \geq 1) = 1 - P(X = 0)$$

$$1 - \frac{\binom{100}{0}\binom{200}{4}}{\binom{300}{4}} = 0.804$$

Example 3-29: Customer Sample-1

A list of customer accounts at a large company contains 1,000 customers. Of these, 700 have purchased at least one of the company's products in the last 3 months. To evaluate a new product, 50 customers are sampled at random from the list. What is the probability that more than 45 of the sampled customers have purchased from the company in the last 3 months?

$$P(X > 45)$$

$$\sum_{x=46}^{50} \frac{\binom{700}{x}\binom{300}{50-x}}{\binom{1000}{50}}$$

Example

Fifty ($N = 50$) computers were manufactured during a certain week. Forty ($K = 40$) operated perfectly, and 10 had at least one defect. A sample of 5 is selected at random ($n = 5$). What is the probability that 4 out of the 5 will operate perfectly?

$$P(X = 4) \quad \frac{\binom{40}{4}\binom{10}{1}}{\binom{50}{5}} = 0.43$$

Poisson Distribution

Binomial distribution for small p & large n → time consuming

→ for very small p & large n ⇒ Poisson distribution "Law of improbable events"

X → Random variable that equals the number of events

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

→ Used to characterize physical situations in which the number of events are during a specific time of interest

→ No memory: Events occurring in one segment of time are independent of the number of events in a non-overlapping segment

→ mean λ must remain constant

$\lambda = E(X) = np$ $\sigma^2 = V(X) = np = \lambda$

Example 3-31: Calculations for Wire Flaws-1

For the case of the thin copper wire, suppose that the number of flaws follows a Poisson distribution with a mean of 2.3 flaws per mm. Find the probability of exactly 2 flaws in 1 mm of wire.

X → number of flaws per mm

$\lambda = 2.3$

$x = 2$

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{2.3^2 e^{-2.3}}{2!} = 0.265$$

Determine the probability of 10 flaws in 5 mm of wire.

X → flaws per 5mm

$x = 10$

$\lambda = np = 2.3(5) = 11.5$ flaws

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{11.5^{10} e^{-11.5}}{10!} = 0.113$$

Determine the probability of at least 1 flaw in 2 mm of wire.

$X \to$ flaw per 2mm

$X \geq 1$

$$\lambda = 2.3 \times 2 = 4.6 \text{ flaws}$$

$$P(X \geq 1) = 1 - P(X < 1)$$

$$1 - \frac{\lambda^x e^{-\lambda}}{x!} = 1 - \frac{4.6^0 e^{-4.6}}{0!} = 0.9899$$

Example

Baggage is rarely lost by Poisson Airlines. A random sample of 1000 flights showed that a total of 300 bags were lost. Find the probability of not losing any bag, exactly one bag is lost.

Poisson distribution is assumed

$$\lambda = \frac{300}{1000} = 0.3$$

$X = 0$

$$P(X=0) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{0.3^0 e^{-0.3}}{0!} = 0.7408$$

$X = 1$

$$P(X=1) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{0.3^1 e^{-0.3}}{1!} = 0.2222$$

The white blood-cell count of a healthy individual can average as low as 6000/mL. To detect white-cell deficiency, a 0.001 mL drop of blood is taken and the number of white cells is found. How many white cells are expected in a healthy individual? If at most two are found, is there an evidence of white-cell deficiency?

$$\lambda = 0.001 \times 6000 = 6$$

↳ 6 white blood cells in avg for a healthy indivisual

$$P(X \leq 2) = \sum_{0}^{2} \frac{\lambda^x e^{-\lambda}}{x!}$$

$$\frac{6^0 e^{-6}}{0!} + \frac{6^1 e^{-6}}{1!} + \frac{6^2 e^{-6}}{2!} = 0.062$$

Lecture 6: Continous Probability Distribution

Continous Random variables

→ Values in uncountable sets

→ Probability Density function

$f(x) \geq 0$

$\int_{-\infty}^{\infty} f(x)\,dx = 1$ [Area under the curve = 1]

$P(a \leq X \leq b) = \int_{a}^{b} f(x)\,dx$ [probability = AUC]

$f(x) = 0$ [no area exactly at x]

Example 4-1: Electric Current

Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes(mA). Assume that the range of X is $4.9 \leq x \leq 5.1$ and $f(x) = 5$. What is the probability that a current is less than 5 mA?

$P(X \leq 5) = \int_{4.9}^{5} f(x)\,dx = \int_{4.9}^{5} 5\,dx = 0.5$

$P(4.95 \leq X \leq 5.1) = \int_{4.95}^{5.1} f(x)\,dx = \int_{4.95}^{5.1} 5\,dx = 0.75$

→ Cumulative Distribution functions

CDF of a continous random variable X is

$$F(X) = P(X \leq x) = \int_{-\infty}^{x} f(u)\,du$$

Example 4-3: Electric Current

For the copper wire current measurement in Exercise 4-1, the cumulative distribution function consists of three expressions.

$$f(x) = \begin{cases} 0 & x < 4.9 \\ 5x - 245 & 4.9 \leq x \leq 5.1 \\ 1 & x \geq 5.1 \end{cases}$$

→ The probability density function of a continous Random variable can be determined from the cumulative distribution function by diffrentiating

$$f(x) = \frac{dF(x)}{dx} = \frac{d}{dx} \int_{-\infty}^{\infty} f(u)\, du$$

Exercise 4-5: Reaction Time

The time until a chemical reaction is complete (in milliseconds, ms) is approximated by this cumulative distribution function:

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-0.01x} & \text{for } 0 \leq x \end{cases}$$

What is the Probability density function?

$$f(x) = \frac{dF(x)}{dx} = \frac{d}{dx} \begin{cases} 0 \\ 1 - e^{-0.01x} \end{cases} = \begin{cases} 0 & x < 0 \\ 0.01 e^{-0.01x} & 0 \leq x \end{cases}$$

What proportion of reactions is complete within 200 ms?

200ms > 0

$$P(X < 200) = F(200) = 1 - e^{-0.01(200)}$$

$$= 0.8647$$

Equivalent of open & Closed bounds

X → continous Random variable, for any x_1 & x_2

$$P(x_1 \leq X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X \leq x_2) = P(x_1 < X < x_2)$$

Expected value & variance

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx \longrightarrow E[h(x)] = \int_{-\infty}^{\infty} h(x) f(x)\,dx$$

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

Example 4-6: Electric Current

For the copper wire current measurement, the PDF is $f(x) = 0.05$ for $0 \leq x \leq 20$. Find the mean and variance.

$$E(X) = \int_0^{20} x(0.05)\,dx = 10$$

$$V(X) = \int_0^{20} x^2(0.05)\,dx - 10^2 = 33.33$$

Normalization: find the constant in f(x) to make it a PDF.
$f(x) = A$ for $0 \leq x \leq 20$

$$\int_0^{20} f(x)\,dx = 1$$

$$\int_0^{20} A\,dx = 1$$

$$A\Big|_0^{20} = 1 \qquad A(20-0) = 1$$

$$A = 0.05$$

Example 4-7:

Let X be the current measured in mA. The PDF is $f(x) = 0.05$ for $0 \le x \le 20$. What is the expected value of power when the resistance is 100 ohms? Use the result that power in watts $P = 10^{-6}RI^2$, where I is the current in milliamperes and R is the resistance in ohms.

Now, $h(X) = 10^{-6}100X^2$.

$$h(x) = 10^{-6}(100)(x^2$$

$$P = E[h(x)] = \int_{-\infty}^{\infty} h(x)\, f(x)\, dx$$

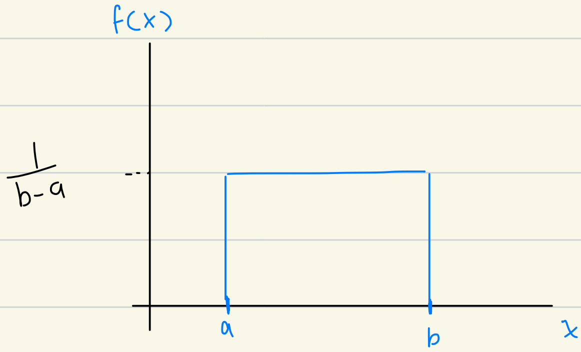
$$10^{-6}(100)(0.05)\int_{0}^{20} x^2\, dx \qquad = \quad 0.0133 \text{ watts}$$

Continous Uniform Distribution

$$f(x) = \frac{1}{b-a} \qquad a \leqslant x \leqslant b$$

$$\mu = \frac{b+a}{2}$$

$$\sigma^2 = \frac{(b-a)^2}{12}$$



Example 4-9: Uniform Current

The random variable X has a continuous uniform distribution on [4.9, 5.1]. The probability density function of X *is* $f(x) = 5$, $4.9 \le x \le 5.1$. What is the probability that a measurement of current is between 4.95 and 5.0 mA?

$$P(4.95 \leqslant x \leqslant 5) = \int_{4.95}^{5} f(x)\, dx = \int_{4.95}^{5} 5\, dx = 0.25$$

$$E(x) = \frac{5.1 + 4.9}{2} = 5 \text{ mA}$$

$$V(x) = \frac{(5.1 - 4.9)^2}{12} = 3.33 \times 10^{-3} \text{ mA}^2$$

→ Cumulative Distribution Function of Uniform Distribution

→ CDF of uniform continous Random distribution is obtained by Integrating the PDF

$$F(x) = \int_a^x \frac{1}{b-a}\,du = \frac{x-a}{b-a}$$

$$F(x) \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \geq b \end{cases}$$

Normal Gaussian Distribution

→ central limit theorm [symmetric]

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad \sigma > 0 \qquad -\infty < \mu < \infty$$

$N(\mu, \sigma^2)$ is used to denote the distribution

→ Bell shape with a single peak at the center of the distribution

Mean = Mode = Median

Empirical rule

$$P(\mu - \sigma < x < \mu + \sigma) = 0.6827$$

$$P(\mu - 2\sigma < x < \mu + 2\sigma) = 0.9545$$

$$P(\mu - 3\sigma < x < \mu + 3\sigma) = 0.9973$$

Example: Battery life

A life test on a large number of type D alkaline batteries revealed that the mean life for a particular use before failure is 19.0 hours. The distribution of lives approximated a normal distribution. The standard deviation was 1.2 hours.

1. About 68% of the batteries failed between what two values?
2. About 95% of the batteries failed between what two values?
3. Virtually all of the batteries failed between what two values?

$\mu=19$, $\sigma= 1.2$

1. $19-1.2 < X < 19+1.2$ $\quad 17.8 < X < 20.2$ $\quad [17.8, 20.2]$

2. $19-2(1.2) < X < 19+2(1.2)$ $\quad 16.6 < X < 21.4$ $\quad [16.6, 21.4]$

3. $19-3(1.2) < X < 19+3(1.2)$ $\quad 15.4 < X < 22.6$ $\quad [15.4, 22.6]$

* A normal Random variable with $\mu=0$ & $\sigma^2=1$ → Standard normal variable (Z)

The cumulative distribution function of a standard normal random variable

$$\phi(Z) = P(Z \leq z)$$

→ Standarizing a normal Random variable

Suppose X is a normal random variable

$$P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) = P(Z \leq z) = \phi z$$

$Z = \frac{x-\mu}{\sigma}$ ⇝ Z value obtained by standarizing X

Example 4-14: Normally Distributed Current-1

Suppose that the current measurements in a strip of wire are assumed to follow a normal distribution with μ = 10 and σ = 2 mA, what is the probability that the current measurement is between 9 and 11 mA?

$\mu = 10$

$\sigma = 2$ mA

$$P(9 < X < 11) = P\left(\frac{9-10}{2} < \frac{X-10}{2} < \frac{11-10}{2}\right)$$

$$P(-0.5 < Z < 0.5)$$

$$P(Z < 0.5) - P(Z < -0.5)$$

$$0.69146 - 0.308538 = 0.38292$$

Determine the value for which the probability that a current measurement is below 0.98.

$$P(X < x) = P\left(\frac{X-10}{2} < \frac{x-10}{2}\right)$$

$$P\left(Z < \frac{x-10}{2}\right) = 0.98$$

$$Z = 2.05$$

$$X = 14.1$$

Example

A tire manufacturer wants to set a minimum mileage guarantee on its new tire. Tests revealed that the mean mileage is 47900 with a standard deviation of 2050 miles and a normal distribution. The manufacturer wants to set the minimum guaranteed mileage so that no more than 4% of the tires will have to be replaced. What minimum guaranteed mileage should the manufacturer announce?

$\mu = 47900$

$\sigma = 2050$

$P = 0.04$

$$Z = \frac{X - \mu}{\sigma}$$

$$Z = -1.75$$

$$-1.75 = \frac{x - 47900}{2050} \qquad X = 44312 \text{ miles}$$

Lecture 7: Joint Probability Distributions

Discrete joint probability distribution

$f_{xy}(x,y) \geq 0$

$\sum_x \sum_y f_{xy}(x,y) = 1$

$f_{xy}(x,y) = P(X=x \; Y=y)$

	x = number of bars of signal strength		
y = number of times city name is stated	**1**	**2**	**3**
4	0.15	0.1	0.05
3	0.02	0.1	0.05
2	0.02	0.03	0.2
1	0.01	0.02	0.25

Joint Probability Density function

$f_{xy}(x,y) \geqslant 0$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\,dx\,dy = 1$$

Example 5-2: Server Access Time

Let the random variable X denote the time until a computer server connects to your machine (in milliseconds), and let Y denote the time until the server authorizes you as a valid user (in milliseconds). X and Y measure the wait from a common starting point ($x < y$). The joint probability density function for X and Y is

$$f_{XY}(x,y) = ke^{-0.001x-0.002y} \text{ for } 0 < x < y < \infty \text{ and } k = 6\cdot 10^{-6}$$

The property that this joint probability density functionintegrates to 1 can be verified by the integral of fXY (x, y) over this region as follows

$$k\int_0^{\infty}\int_0^{\infty} e^{-0.001x-0.002y}\,dy\,dx = k\int_0^{\infty}\left[\int_x^{\infty} e^{-0.002y}\,dy\right] e^{-0.001x}\,dx$$

$$= k\int_0^{\infty}\left[\frac{e^{-0.002x}}{0.002}\right] e^{-0.001x}\,dx$$

$$= 0.003\int_0^{\infty} e^{-0.003x}\,dx$$

$$= 0.003\,\frac{e^{-0.003(\infty)}}{0.003} = 0.003\left(\frac{1}{0.003}\right) = 1$$

The probability that X< 1000 and Y< 2000 is determined as the integral over the darkly shaded region in Fig. 5-5.

$$P(X \leq 1000, Y \leq 2000) = \int_0^{1000} \int_x^{2000} f(x,y)\, dy\, dx$$

$$k \int_0^{1000} \left[\int_x^{2000} e^{-0.002y}\, dy \right] e^{-0.001x}\, dx$$

$$k \int_0^{1000} \left(\frac{e^{-0.002x} - e^{-4}}{0.002} \right) e^{-0.001x}\, dx$$

$$= 0.003 \left[e^{-0.003x} - e^{-4} e^{-0.001x} \right]$$

$$= 0.003 \left[\left(\frac{1 - e^{-3}}{0.003} \right) - e^{-4} \left(\frac{1 - e^{-1}}{0.001} \right) \right] = 0.915$$

Marginal Probability Distributions (Discrete)

$$f_X(x) = \sum_y f(x,y)$$

$$f_Y(y) = \sum_x f(x,y)$$

y = Response time(nearest second)	x = Number of Bars of Signal Strength			
	1	2	3	f(y)
1	0.01	0.02	0.25	0.28
2	0.02	0.03	0.20	0.25
3	0.02	0.10	0.05	0.17
4	0.15	0.10	0.05	0.30
f(x)	0.20	0.25	0.55	1.00

Marginal Probability Density function (continous)

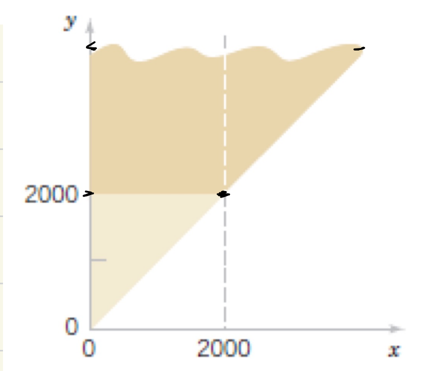
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy \qquad -\infty < x < \infty$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx \qquad -\infty < y < \infty$$

Example 5-4: Server Access Time

For the random variables that denotes times in Example 5-2, find the probability that Y exceeds 2000 milliseconds.

Integrate the joint PDF directly using the picture to determine the limits.



$$P(Y>2000) = \int_0^{2000}\int_{2000}^{\infty} f(x,y)\,dy\,dx + \int_{2000}^{\infty}\int_{x}^{\infty} f(x,y)\,dy\,dx$$

or $P(Y>2000) = \int_0^{y} f(x,y)\,dx$

$$k\int_0^{y} e^{-0.001x-0.002y}\,dx$$

$$ke^{-0.002y}\int_0^{y} e^{-0.001x}\,dx$$

$$ke^{-0.002y}\left[\frac{e^{-0.001x}}{-0.001}\Big|_0^{y}\right]$$

$$ke^{-0.002y}\left[\frac{1-e^{-0.00y}}{0.001}\right]$$

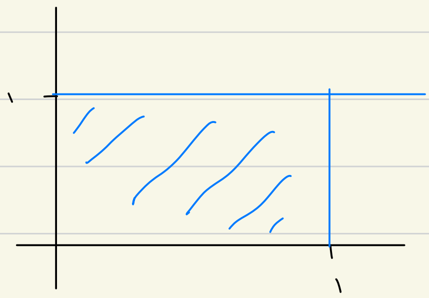
$$\frac{6\times10^{-6}}{0.001}\,e^{-0.002y}\left(1-e^{-0.001y}\right)$$

Cashews & Nuts Example (Continuous)

A nut company markets cans of deluxe mixed nuts containing almonds, cashews, and peanuts. The net weight of the can is 1 lb, but the weight contribution of each type of nut is random. Because the three weights sum to 1, a joint probability model for any two gives all necessary information about the weight of the third type. Let X= the weight of almonds in a selected can and Y= weight of cashews. The joint PDF for (X,Y) was suggested to be

$$f(x,y)=\begin{cases}24xy & 0\le x\le 1, 0\le y\le 1, x+y\le 1\\ 0 & \text{otherwise}\end{cases}$$

$$\int_0^1\int_0^{1-x} 24xy\,dy\,dx$$



$$\int_0^1 \frac{24xy^2}{2}\Big|_0^{1-x}\,dx$$

$x+y=1$

$y=1-x$

$$\int_0^1 12x(1-x)^2 = \underline{\underline{1}} \Rightarrow \text{PDF}$$

$$f_X(x)=\int_0^{1-x} 24xy\,dy = \begin{cases}12x(1-x)^2 & 0\le x\le 1\\ 0 & \text{otherwise}\end{cases}$$

Marginal Probability

$$f_Y(y)=\int_0^{1-y} 24xy\,dx = \begin{cases}12y(1-y)^2 & 0\le y\le 1\\ 0 & \text{otherwise}\end{cases}$$

Independence $f_{XY}(xy)=f_X(x)\,f_Y(y) \Rightarrow$ Independent

$$24xy \overset{?}{=} \left[12x(1-x)^2\right]\left[12y(1-x)^2\right]$$

$$24xy \neq 144xy(1-x)^2(1-y)^2$$

dependent

$x+y=1$

$1-x-y$

Expected Cost

$$h(x,y) = (1)x + (1.5)y + 0.5(1-x-y)$$

$$x + 1.5y + 0.5 - 0.5x - 0.5y$$

$$0.5 + 0.5x + y$$

$$h(X,Y) = \int_0^1 \int_0^{1-x} (0.5 + 0.5x + y)\, 24xy \, dy \, dx$$

Covariance:

$$\mu_x = \mu_y = \int_0^1 12x^2(1-x)^2 \, dx = 2/5$$

$$\sigma_x = \sigma_y = \sqrt{\int_0^1 12x^3(1-x)^2 \, dx - \left(\frac{2}{5}\right)^2} = 1/5$$

$$E(X,Y) = \sigma_{xy} = \int_0^1 \int_0^{1-x} xy\,(24xy)\, dy\, dx$$

Correlation

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

lecture 7 : Tests of Hypothesis

→ hypothesis is a statement about a population developed for the purpose of testing

→ Symmetric procedure based on sample evidence & probability theory to determine whether the hypothesis is a reusonable statement

1. state the null (H_0) & alternate (H_1) hypothesis
2. select a level of significance
3. Identify (compute) the test statistic
4. Formulate a disicion rule
5. Reject or donot reject H_0 & accept H_1

1. State the Null (H_0) & alternate (H_1) hypothesis

→ the null hypothesis is a statement that is not rejected if our sample data fail to provide convincing evidence that it is false

not rejecting doesn't mean that H_0 is true (only failed to disprove H_0)

→ Alternate hypothesis is a statement that is accepted if the sample data provide enough evedince that the null hypcthesis is false

Null always has =

2. Select a level of significance

→ Probability of rejecting the null hypothesis when it's true

→ Use 0.05 unless stated otherwise

Type I error → Rejecting null hypothesis when it's true α

Type II error → Accepting null hypothesis when it's false β

3. Compute the test-statistic

→ Value determined from sample information, used to determine whether to reject or accept Ho

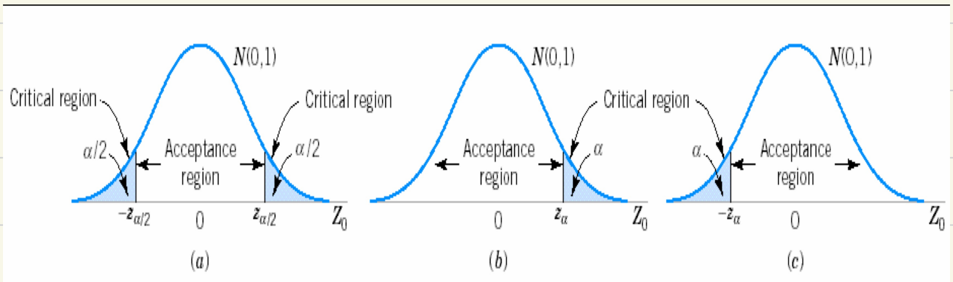
$$z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

z → standard normal distribution

4- Formulate the Desicion Rule

→ Statement of the conditions under which the Ho is rejected or not rejected

Critical value is the dividing point between rejecting & not



Tests of significance

one tailed [b, c]

lower $H_0 : \mu = X$

$H_1 : \mu < X$

Upper $H_0 : \mu = X$

$H_1 : \mu > X$

Two tailed [a]

$H_0 : \mu = X$

$H_1 : \mu \neq X$

P-value

The P-value is the smallest level of significance at which Ho would be rejected when a specified test procedure is used on a given data set

P-value $\leq \alpha$ Reject Ho at level α

P-value $> \alpha$ Donot Reject Ho at level α

5 Make a desicion

→ Depending on the z-value either reject H_0 or donot reject it

Test on Mean with known σ

$H_0: \mu = \mu_0$

$$z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$

$H_1: \mu > \mu_0$ $\quad z_0 > z_\alpha$

$H_1: \mu < \mu_0$ $\quad z_0 \leq -z_\alpha$

$H_1: \mu \neq \mu_0$ $\quad z_0 \geq z_{\alpha/2}$ or $z_0 \leq -z_{\alpha/2}$

- Aircrew escape systems are powered by a solid propellant. The burning rate of this propellant is an important product characteristic. Specifications require that the mean burning rate must be 50 centimeters per second. We know that the standard deviation of burning rate is 2 cm/s. The experimenter decides to specify a type I error probability or significance level of 0.05 and selects a random sample of n=25 and obtains a sample average burning rate of 51.3 cm/s. What conclusions should be drawn?

$\sigma = 2 cm/s$

$\alpha = 0.05$

$n = 25$

$\bar{x} = 51.3 cm/s$

1. $H_0 : \mu = 50 cm/s$

   $H_1 : \mu \neq 50 cm/s$

2. $\alpha = 0.05$

3. $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \dfrac{51.3 - 50}{2/\sqrt{25}} = 3.25$

4. reject H_0 if $\rightarrow$ $H_1 : \mu \neq \mu_0$ $\quad z_0 \geq z_{\alpha/2}$ or $z_0 \leq -z_{\alpha/2}$

$z_{\alpha/2} = z_{0.025}$ $\quad -z_{0.025} = -1.96$ $\quad z_{0.025} = 1.96$

$3.25 \geq 1.96$ Reject H_0

5. Since $z_0 = 3.25 \geq 1.96$ we reject $H_0 : \mu = 50$ at the 0.05 level of significance

$$\Rightarrow \text{P-value} = \begin{cases} 2[1-\Phi(|z_0|)] & \text{for two tailed test} \quad H_0: \mu = \mu_0 \quad H_1: \mu \neq \mu_0 \\ 1 - \Phi(z_0) & \text{for upper tailed test} \quad H_0: \mu = \mu_0 \quad H_1: \mu > \mu_0 \\ \Phi(z_0) & \text{for lower tailed tes} \quad H_0: \mu = \mu_0 \quad H_1: \mu < \mu_0 \end{cases}$$

P-value $= 2[1-\Phi(|z_0|)]$

$1 - 0.999423 = (0.000577) \times 2$ $\quad P = 0.0012$

$\alpha >$ P-value

$0.025 > 0.0012$ $\Rightarrow$ Ho will be rejected any $\alpha > P = 0.0012$

- The increased availability of light materials with high strength has revolutionized the design and manufacture of golf clubs, particularly drivers. Clubs with hollow heads and very thin faces can result in much longer tee shots, especially for players of modest skills. This is due partly to the "spring-like effect" that the thin face imparts to the ball. Firing a golf ball at the head of the club and measuring the ratio of the outgoing velocity of the ball to the incoming velocity can quantify this spring-like effect. The ratio of velocities is called the coefficient of restitution of the club. An experiment was performed in which 15 drivers produced by a particular club maker were selected at random and their coefficients of restitution measured. In the experiment the golf balls were fired from an air cannon so that the incoming velocity and spin rate of the ball could be precisely controlled. It is of interest to determine if there is evidence (with $\alpha = 0.05$) to support a claim that the mean coefficient of restitution exceeds 0.82. The observations follow:
- **0.8411 0.8191 0.8182 0.8125 0.8750 0.8580 0.8532 0.8483 0.8276 0.7983 0.8042 0.8730 0.8282 0.8359 0.8660**
- The sample mean is 0.83725 and $s = 0.02456$. The normal probability plot of the data in Fig. 9-9 supports the assumption that the coefficient of restitution is normally distributed. Since the objective of the experimenter is to demonstrate that the mean coefficient of restitution exceeds 0.82, a one-sided alternative hypothesis is appropriate.

$n = 15 \leq 30 \to t$

$\alpha = 0.05$

$\bar{x} = 0.83725$

$s = 0.02456$

1. $H_0 : \mu = 0.82$

   $H_1 . \ \mu > 0.82$

2 $\alpha = 0.05$

3. $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n}}$ $\quad t = \dfrac{0.83725 - 0.82}{\frac{0.02456}{\sqrt{15}}} = 2.72$

4. Reject Ho if $H_1 : \mu > \mu_0$ $\quad t_0 \geq t_{\alpha, n-1}$

$2.72 \geq 1.761$ $\Longrightarrow$ Reject Ho at level of significance $= 0.05$

5. Reject $H_0 : \mu = 0.82$ at the level of significance $\alpha = 0.05$, therfore the coefficient of restitution exceeds 0.82

Lecture 8: Descriptive Statistics

Measures of central tendancy [Arithmatic Mean]

$$\mu = \sum_{i=1}^{N} x f(x) = \frac{\sum x_i}{N}$$ → for a finite population with N measurements

$$\bar{x} = \frac{\sum x_i}{N}$$ → Sample mean is a reasonable estimate of the population mean.

→ Geometric Mean (screen sizes) $$\bar{x}_g = \left[\prod_{i=1}^{n} x_i\right]^{1/n}$$

→ Harmonic Mean (mole / mass fractions) $$\bar{x}_h = \frac{n}{\sum \frac{1}{x_i}}$$

→ other Types of means:-

1. log mean temperature difference (LMTD) [log based averaging]
2. Sauter mean drop size in liq-liq dispersions [volume to SA ratio]

Median

- The median is very sensative to outliers or extreme values in the data
  Rigorous estimator of central tendancy
- 50% of the values are above, 50% are below [sort the data in ascending order]

$$\tilde{x} = \begin{cases} x_{(n+1)/2} & , n = \text{odd} \\ \dfrac{x_{(n/2)} + x_{n/2+1}}{2} & , n = \text{even} \end{cases}$$

- it's the 50^{th} percentile or second quartile
- Best obtained using a cumulative distribution

Mode

- The value of the most frequantly encountered observation [most frequant value]
- The maximum of the distribution

Measures of Dispersion : Range

- Range is the difference between the maximum & the minimum values of the random variable we are interested in [information about the spread of the data]

$$\text{Range} = R = x_{max} - x_{min}$$

Variance & standard deviation

Sample variance $$s^2 = \frac{\sum (x_i - \bar{x})^2}{n-1} = \frac{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}{n-1}$$

population variance $$\sigma^2 = \frac{\sum (x_i - \mu)^2}{N}$$

Sample standard deviation $$s = +\sqrt{s^2}$$

population standard deviation $$\sigma = +\sqrt{\sigma^2}$$

Range = 13.6 − 12.3 = 1.3

Table 6-1 Calculation of Terms for the Sample Variance and Sample Standard Deviation

i	x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
1	12.6	−0.4	0.16
2	12.9	−0.1	0.01
3	13.4	0.4	0.16
4	12.3	−0.7	0.49
5	13.6	0.6	0.36
6	13.5	0.5	0.25
7	12.6	−0.4	0.16
8	13.1	0.1	0.01
	104.0	0	1.6

Sample variance. $s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1} = \frac{1.6}{8-1} = 0.2286$

$\bar{x} = \frac{\Sigma x_i}{n} = \frac{104}{8} = 13$

Sample standard deviation $s = \sqrt{s^2} = 0.478$

population variance $\sigma^2 = \frac{\Sigma(x-\mu)^2}{N} = \frac{1.6}{8} = 0.2$

population SD $\sigma = \sqrt{\sigma^2} = 0.447$

Coefficient of Variation

The coefficient of variation [CV] is the ratio of standard deviation to the arithmatic mean

$$CV = \frac{s}{\bar{x}} \times 100\%$$

* A good measure for comparing different values of means & standard deviations [relative]

Measure of Symmetry · Skewness

measure of Dispersion give some values about the spread of a distribution but they don't provide any info about the shape of the distribution around the mean or median.

Coefficient of skewness (sk) is defiened to alleviate such lack of info

$$sk = \frac{3(\text{mean} - \text{median})}{\text{standard deviation}} = \frac{3(\bar{x} - \tilde{x})}{s}$$

Symmetric → mean = median

if one mode exists (unimodal)

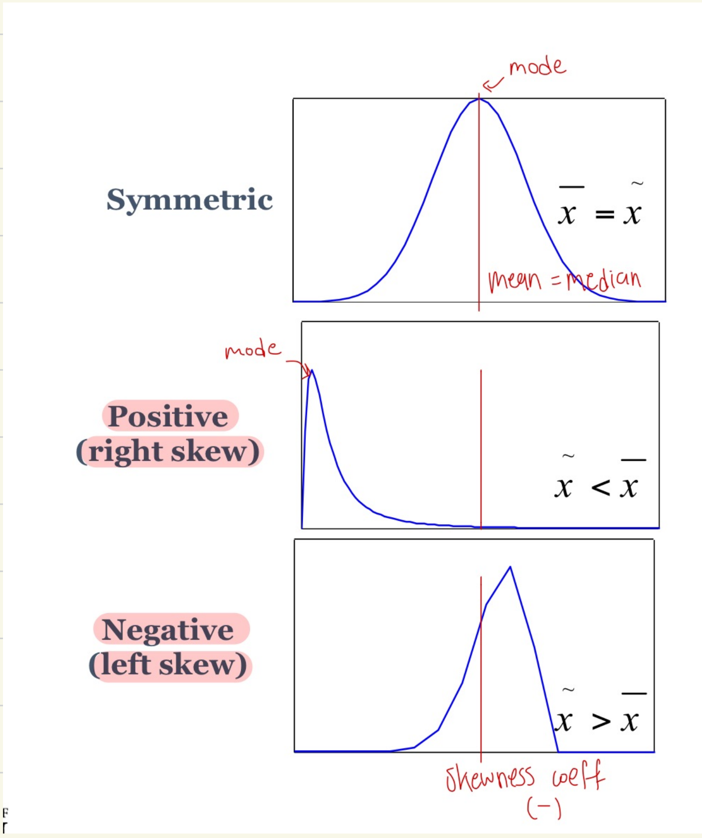
mean = median = mode

Skewed to the right →

mode < median < mean

skewed to the left →

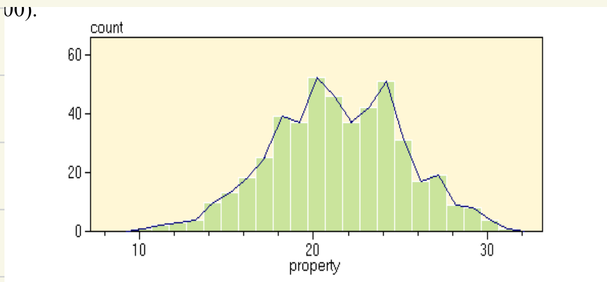
mode > median > mean



Histogram plots [frequancy Distribution]

Plots of frequancy vs the property of intrest

Good for display of the shape (trend) of the data for relatively large samples (n>100)



→ to Generate a frequancy distribution :

- Divide the range of data into intervals [Class intervals, cells or bins]
- Choose a number of bins ⇒ use the square root of the number of observations (n)
- find the frequancy of observations in each bin

Relative frequancy (normalized) : the observed frequency in each bin divided by the total number of observations

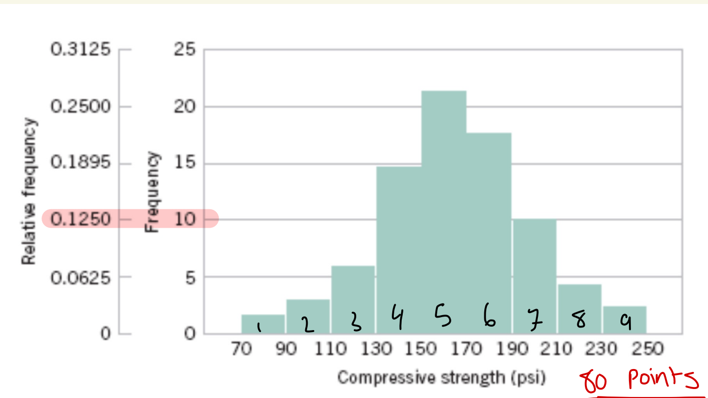
Cumulative frequancy : the height of each bar is the total number of observations that are less than or equal to the upper limit of the bin

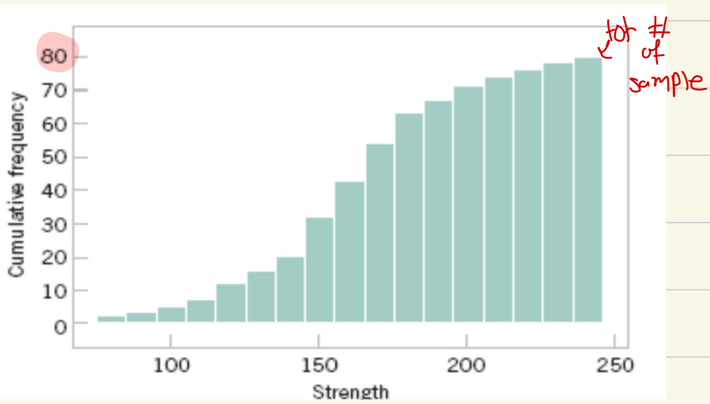
Bin selection:

→ square root of the number of observations

→ Freedman - Diaconis Rule:

$$\text{Bin size} = 2 \cdot \underbrace{IQR}_{\text{Inter quartile Range}} (\underbrace{X}_{\text{Data}}) \cdot \underbrace{n^{-\frac{1}{3}}}_{\text{number of observations}}$$





number of Intervals = $\sqrt{80} \simeq 9$

width of Interval = $\dfrac{\text{Range}}{\#\text{ of intervals}} = \dfrac{250-70}{9} = 20$

Relative frequancy = $\dfrac{\text{frequancy}}{\text{tot }\#\text{ of data Points}} = \dfrac{10}{80} = 0.125$

* In cumulative frequancy ⇒ Addition of Relative frequancy only

Frequancy polygons:

- same as histograms
- smoother alternative to histograms
- joining midpoints of the histogram bars with lines
- Area below histograms & frequancy polygons are equal

histograms are more preferable.

Box plots

- The box plot is a graphical display that simultaneously describes several important features ⇒ Center, spread, departure from symmetry, & identification of observations that lie unusually far from the bulk of the data
  - whisker [max, min]
  - outlier
  - Extreme oultlier

1. Arrange the data in Ascending order
2. q_2 = Median of Data
3. q_1 & q_3 = Median of Data on either side of first Median

4. Check for outliers $[Q_1 - 1.5\ IQR,\ Q_3 + 1.5\ IQR]$

$IQR = Q_3 - Q_1$

Ex: 18, 34, 76, 29, 15, 41, 46, 25, 54, 38, 20, 32, 43, 22

1. 15, 18, 20, 22, 25, 29, 32, 34, 38, 41, 43, 46, 54, 76

min q_1 q_3 max   max = outlier

$q_2 = 33$

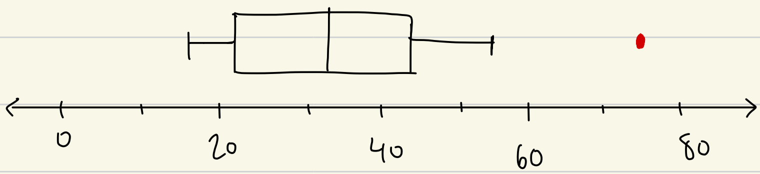
$[Q_1 - 1.5\,IQR,\ Q_3 + 1.5\,IQR]$

$[-9.5,\ 74.5]$

$IQR = 43 - 22 = 21$

→ min is inside Range

→ max is an outlier



Probability plots:

- Determines whether the sample data conforms to a hypothesized distribution based on a subjective visual examination of the data

→ Uses special graph paper [probability paper]

1. Sort the values (x) in Ascending order
2. Add a column (j) to the left assigning a number starting from 1 to n next to each sorted value
3. Calculate the cumulative frequancy $y = (j - 0.5)/n$
4. Plot
5. Draw a line through the data using 25^{th} & 75^{th} percentile points
6. If the line passes nicely through the points ⇒ distribution describes the data well.

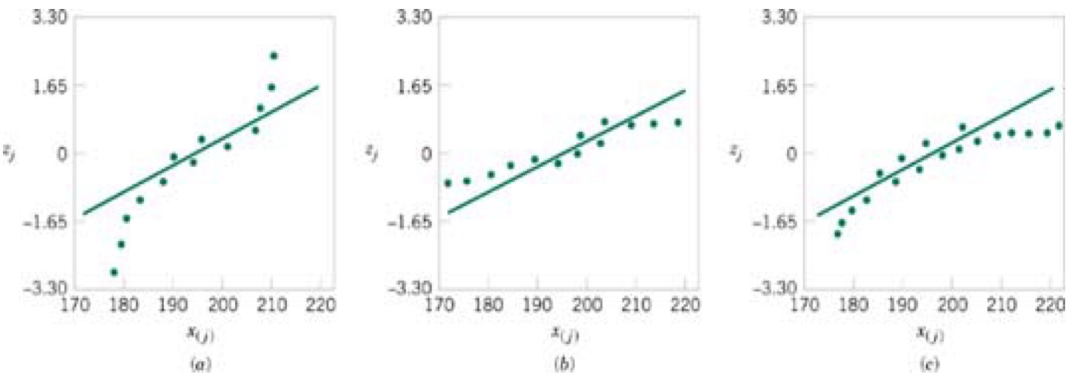


FIGURE 6-21 Normal probability plots indicating a nonnormal distribution.

a. Light-tailed distribution.

b. Heavy-tailed distribution.

c. A distribution with positive (or right) skew.

A. when a sample is selected from the light-tailed distribution, the smallest & largest distributions will not be as extreme; observations on the left side will fall below the line, observations on the right side will fall above the line ⇒ S-shaped probability line

B. Heavy tailed distribution ⇒ S-shaped probability line but observations on the left will be above the line, observations on the left will be below the line.

C. Points on both ends of the plot will fall below the line ⇒ curved shape. Both smallest & largest observations are larger than expected in a sample from a normal distribution

Stem & leaf Diagram

A visual display of Data set $x_1, x_2, \dots x_n$ where each number consists of atleast 2 digits

Lecture 9 : Linear Regression

→ objective is to build a model of a set of data : - prediction
- Interpolation / extrapolation
- optimization

→ The parameters: Regression Coefficients

→ Regressor (predictor) : set of independent variables

→ Response : dependent variable

Sample :

Observation Number	Hydrocarbon Level x (%)	y (%)
1	0.99	90.01
2	1.02	89.05
3	1.15	91.43
4	1.29	93.74
5	1.46	96.73
6	1.36	94.45

Each observation Y can be described by the model

$$\hat{y} = \beta_0 + \beta_1 x + \varepsilon$$

↳ Random error term

* Assumptions for ε : zero mean value
  variance is σ^2

→ Least squares method is used to estimate the parameters β_0 & β_1 by minimizing the sum of squares of the vertical deviations

The Sum of the Squares of the Errors SSE (residuals)

$$L = \sum \varepsilon^2 = \sum (y - \hat{y})^2 = \sum (y_i - \beta_0 - \beta_1 x)^2$$

* for the function L to be minimum → it's derivatives with respect to all parameters must be zero

→ $\beta_0 n + \beta_1 \sum x = \sum y$

→ $\beta_0 \sum x + \beta_1 \sum x^2 = \sum yx$

→ $$\beta_1 = \frac{\sum yx - \left[\frac{\sum y \sum x}{n}\right]}{\sum x^2 - \frac{\left[\sum x\right]^2}{n}} = \frac{S_{xy}}{S_{xx}}$$

$$\beta_0 = \bar{y} - \beta_1 \bar{x}$$

→ $\bar{y} = \frac{\sum y}{n}$ → $\bar{x} = \frac{\sum x}{n}$

→ $SSE = \sum \varepsilon^2 = \sum (y_i - \hat{y}_i)^2$ Error sum of squares

$SST > SSE$

→ $SST = \sum (y_i - \bar{y})^2$ Total sum of squares

→ Unbaised estimator

$$\sigma^2 = \frac{SSE}{n-2}$$

→ Confidence limits

$$\beta_1 - t_{\alpha/2, n-2}\sqrt{\frac{\sigma^2}{S_{xx}}} \leq \beta_1 \leq \beta_1 + t_{\alpha/2, n-2}\sqrt{\frac{\sigma^2}{S_{xx}}}$$

$$\beta_0 - t_{\alpha/2, n-2}\sqrt{\sigma^2\left[\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right]} \leq \beta_0 \leq \beta_0 + t_{\alpha/2, n-2}\sqrt{\sigma^2\left[\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right]}$$

Adequacy of a Regression Model

→ Fitting a Regression Model requires several Assumptions:

- Errors are uncorrelated Random variables with mean zero
- Errors have constant variance
- Errors are normally distributed

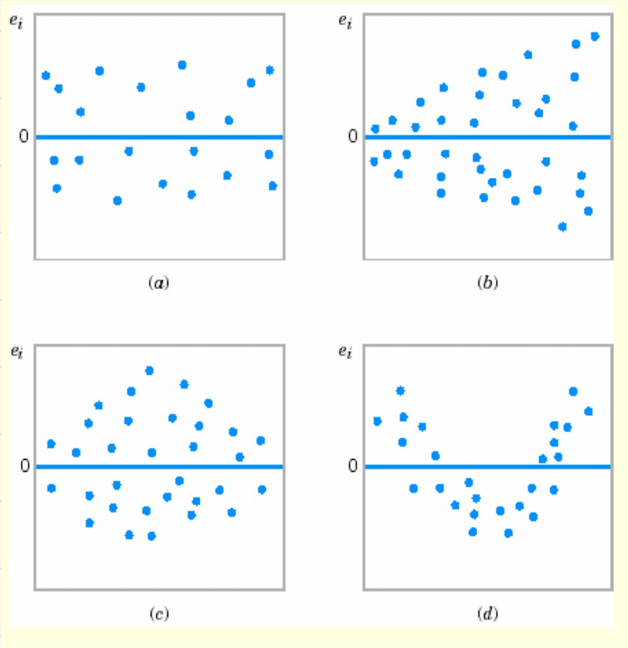
→ Residual Analysis

The residuals from a regression model are $\varepsilon = y_i - \hat{y}_i$

Actual observation el ↳ corresponding fitted value

a → adequate (Random)

b, c, d → Inadequate (has specific shape)



→ coefficient of determination

$$R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$$

$$0 \leq R^2 \leq 1$$

$$-1 \leq R \leq 1$$

Power law

$$y = ax^b$$

$$\ln y = \ln a + b \ln x$$

$$y' = \beta_0 + \beta_1 x'$$

Exponential

$$y = ae^{bx}$$

$$\ln y = \ln a + bx$$

$$y' = \beta_0 + \beta_1 x$$

Saturation

$$y = \frac{ax}{1+bx}$$

$$\frac{1}{y} = \frac{b}{a} + \frac{1}{ax}$$

$$y' = \beta_0 + \beta_1 x'$$

Ex. x	y	xy	x^2	y'	$(y-y')^2$	$(y-\bar{y})^2$
0.99	90.01	89.109	0.9801	89.225	0.6162	6.5448
1.02	89.05	90.831	1.0404	89.678	0.3943	12.378
1.15	91.43	105.14	1.3225	91.638	0.0432	1.2957
1.29	93.74	120.92	1.6641	93.749	8.1×10^{-5}	1.3728
1.46	96.73	141.22	2.1316	96.313	0.1738	17.319
1.36	94.45	128.45	1.8469	94.805	0.126	3.5407
Sum 7.27	555.41	675.68	8.9883	555.41	1.3583	42.452

$$\bar{x} = \frac{\Sigma x}{n} \qquad \bar{y} = \frac{\Sigma y}{n}$$

$$\bar{x} = \frac{7.27}{6} = 1.21166 \qquad \bar{y} = \frac{555.41}{6} = 92.5683$$

$$\beta_1 = \frac{\Sigma yx - \left[\frac{\Sigma y \, \Sigma x}{n}\right]}{\Sigma x^2 - \frac{[\Sigma x]^2}{n}} = \frac{S_{xy}}{S_{xx}} \qquad \beta_0 = \bar{y} - \beta_1 \bar{x}$$

$$\beta_1 = \frac{675.68 - \left[\frac{555.4 \times 7.27}{6}\right]}{(8.9883) - \frac{(7.27)^2}{6}} = 15.08$$

$$\beta_0 = 92.5683 - (15.08)(1.21166) = 74.2964$$

$$y' = 74.2964 + 15.08x$$

$$R^2 = 1 - \frac{SSE}{SST} \qquad SSE = \Sigma(y - \hat{y}_i)^2 = 1.3583$$

$$SST = \Sigma(y_i - \bar{y})^2 = 42.452$$

$$R^2 = 1 - \frac{1.3583}{42.452} = 0.968$$

$R = 0.9838$

→ confidence limits

$$\beta_1 - t_{\alpha/2, n-2}\sqrt{\frac{\sigma^2}{S_{xx}}} \leq \beta_1 \leq \beta_1 + t_{\alpha/2, n-2}\sqrt{\frac{\sigma^2}{S_{xx}}}$$

$$15.08 - 2.776\sqrt{\frac{0.339575}{0.17948}} \leq \beta_1 \leq 15.08 + 2.776\sqrt{\frac{0.339575}{0.17948}}$$

$$11.26 \leq \beta_1 \leq 18.89$$

$$\sigma^2 = \frac{SSE}{n-2} = \frac{1.3583}{6-2} = 0.339575$$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n} = 8.9883 - \frac{(7.27)^2}{6} = 0.17948$$

$$\beta_0 - t_{\alpha/2, n-2}\sqrt{\sigma^2\left[\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right]} \leq \beta_0 \leq \beta_0 + t_{\alpha/2, n-2}\sqrt{\sigma^2\left[\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right]}$$

$$74.2964 - 2.776\sqrt{0.339575\left(\frac{1}{6} + \frac{1.21166^2}{0.17948}\right)} \leq \beta_0 \leq \ldots$$

$$69.622 \leq \beta_0 \leq 78.96$$