mathematical methods in chemical engineering

First semester 2025-2026

د.رياض الشوابكة

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Lec(1)

* Differential equations:
- ordinary differential equations (ODEs)
- Partial differential equations (PDEs)

* Variables:
- dependent
- independent

* $y = f(x)$, y : dependent on X
; X : independent

* $y = 3x^2 + x$
y changes according to X change

diff form * $\frac{dy}{dx} + y = x$, y changes for X
dx ind. (always!)
only one independent Variable → $\boxed{\text{(ODEs)}}$

* $\frac{dT}{dt} = \alpha \frac{\partial^2 T}{\partial x^2}$
2 independent Varibles → (PDEs)

* System:

- $\frac{dy}{dx} + z = 3$
- $\frac{dz}{dx} + y = 6$

two dep. Varibles
1 ind. varible → (ODEs)

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ODEs :

* 1^{st} order

- $\frac{dy}{dx} + 3 = e^x \rightarrow \boxed{\frac{dy}{dx} = f(x)}$
- $\frac{dy}{dx} + y = e^x \rightarrow \frac{dy}{dx} = f(x,y)$
- $\frac{dy}{dx} + y = 3 \rightarrow \frac{dy}{dx} = f(y)$

* 2^{nd} order

- $\frac{d^2y}{dx^2} + f(x,y)\frac{dy}{dx} + g(x,y)y = a(x,y)$
- $\frac{d^2y}{dx^2} = 6$
- $\frac{d^2y}{dx^2} + y = 12$
- $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

* higher order

$$\frac{d^ny}{dx^n} + \frac{d^{n-1}y}{dx^{n-1}} \dots = a(x,y)$$

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x$$

* system of equations:

$$\frac{dy}{dx} + yz = x$$

$$\frac{dz}{dx} + y = 3$$

solve both

MADAR

* constant coeff :

$\alpha \frac{dy}{dx} + \beta y = \gamma$, α, β, γ are constants

$$3 \frac{dy}{dx} + 2y = 5$$

* non constant coeff :

$$x \frac{dy}{dx} + 6y = 2$$

* linear vs non linear (ODEs)

linear:

$y'' - y = e^x$

non linear:

$(y')^2 = e^x$ Power

$y y'' - y = e^x$ the dep. var multiplyed by a var.

$y' + y = \boxed{e^y}$ the dep. var. is not linear

based on y dependent ←

* Techniques of solution of 1^{st} order (ODEs)

1) direct integration

$$y' = f(x) \rightarrow \frac{dy}{dx} = f(x) \rightarrow \int dy = \int f(x)\,dx$$
$$y = g(x)$$

$$y' = f(y) \Rightarrow \frac{dy}{dx} = f(y) \rightarrow \int \frac{dy}{f(y)} = \int dx$$

$$y' = f(x,y) \rightarrow \frac{dy}{dx} = f(x)g(y) \rightarrow \int \frac{dy}{g(y)} = \int \frac{dx}{f(x)}$$

example (1):

$$y' = e^{5x} \rightarrow \int dy = \int e^{5x}\,dx$$
$$y = \frac{e^{5x}}{5} + C$$

$$y' + y = 0 \rightarrow \frac{dy}{dx} + y = 0 \rightarrow \frac{dy}{dx} = -y$$

$$\int \frac{dy}{y} = \int -dx \rightarrow e^{\ln y} = e^{-x + c_1}$$
$$y = e^{[-x + c_1]}$$
$$y = e^{-x} \cdot e^{c_1}$$
$$y = C_2 e^{-x}$$

$$y' + y = 10 \rightarrow \frac{dy}{dx} + y = 10 \rightarrow \frac{dy}{dx} = 10 - y$$

$$\int \frac{dy}{10-y} = \int dx \rightarrow -\ln(10-y) = x + c_1$$
$$\ln(10-y) = -x + C_2$$
$$10 - y = e^{-x} C_3$$
$$y = 10 - C_3 e^{-x}$$

* Separation of varibles

$\frac{dy}{dx} = f(x)g(y) \rightarrow \frac{dy}{g(y)} = \frac{dx}{f(x)}$

$\frac{dy}{dx} = \frac{x^2}{y^2} \rightarrow y^2 dy = x^2 dx \; , \; \frac{y^3}{3} = \frac{x^3}{3} + C_1$

2] * exact solution of ODE

$M(x,y) + N(x,y)\frac{dy}{dx} = 0$

example:

$(2x^2y) + (xy^2+1)\frac{dy}{dx} = 0$

$\rightarrow M(x,y) = 2x^2y$
$\rightarrow N(x,y) = xy^2+1$

* if $M_y = N_x$

$\left(\frac{\partial M}{\partial y}\right) = \left(\frac{\partial N}{\partial x}\right) \rightarrow$ exact ODE

$M_y \neq N_x \longrightarrow$ Not-exact ODE

1] direct intigration.

2] exact Differential equation of (ODEs)

$$dy/dx = f(x,y) \rightarrow \underbrace{M(x,y)}_{M}\boxed{dx} + \underbrace{N(x,y)}_{N}\boxed{dy} = 0$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightsquigarrow 1^{st}$ ODEs is exact *

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \rightsquigarrow 1^{st}$ ODEs is not exact *

How to obtain solution for exact DE :.

a) intigrate M with respect to x $\int M\,dx + h(y)$ (constant of y)

b) differentiate $\boxed{a}$ w.r.t y

c) equate $\boxed{b}$ w.r.t N

d) $\int h'(y)\,dy \rightsquigarrow$ solution in $\boxed{a}$

example section :.

$\rightarrow (\underbrace{y\cos x + 2xe^y}_{M}) + (\underbrace{\sin x + x^2e^y + 2}_{N})\frac{dy}{dx} = 0$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= \cos x + 2xe^y \\ \frac{\partial N}{\partial x} &= \cos x + 2xe^y \end{aligned}\right] \text{(exact)}$$

1) $\int (y\cos x + 2xe^y)\,dx$

$= y\sin x + x^2e^y + h(y)$

2) $\sin x + x^2e^y + h'(y)$

3) $\sin x + x^2e^y + h'(y) = N$

$h'(y) = 2 \rightsquigarrow h(y) = 2y \rightsquigarrow$ Solu :. $(y\sin x + x^2e^y + 2y)$

$\rightarrow (\underbrace{2xy+9x^2}_{M}) + (\underbrace{2y+x^2+1}_{N})\,y' = 0$

a) $\frac{\partial M}{\partial y} = 2x \quad , \frac{\partial N}{\partial x} = 2x \quad \rightsquigarrow$ exact

1) $\int (2xy + 9x^2)\,dx$

$= x^2 y + 3x^3 + h(y)$

2) $x^2 + h'(y)$

3) $x^2 + h'(y) = 2y + x^2 + 1$

$h'(y) = 2y + 1$

4) $h(y) = y^2 + y$

5) $x^2 y + 3x^3 + y^2 + y$

$\rightarrow \frac{dy}{dx} = \frac{-(x^3 - 3xy^2)}{3x^2y + y^3}$

$(3x^2y + y^3)\,dy = -(x^3 + 3xy^2)\,dx$

$(\underbrace{3x^2y + y^3}_{N})\,dy + (\underbrace{x^3 + 3xy^2}_{M})\,dx = 0$

$\frac{\partial M}{\partial y} = +6xy \quad , \frac{\partial N}{\partial x} = 6xy \rightsquigarrow$ exact

⋮

3) integrating factor:-

$dy/dx + P(x)y = Q(x)$ → (any poly) → (any poly)

- $\mu = e^{\int P(x)\,dx}$
- $y = \frac{1}{\mu}\int \mu\, Q(x)\,dx$

example section:-

$\rightarrow \quad dy/dx + 2y = e^{-2x}$

$P(x) = 2 \quad , Q(x) = e^{-2x}$

$\rightarrow \mu = e^{\int P(x)dx}$

$= e^{2x}$

$\rightarrow y = \frac{1}{e^{2x}}\int e^{2x}e^{-2x}\,dx$

$= \frac{1}{e^{2x}}\int dx$

$= e^{-2x}(x + c)$

$\rightarrow dy/dx + xy = \frac{1}{x}$

$P(x) = x \quad , Q(x) = \frac{1}{x}$

$\rightarrow \mu = e^{\int x\,dx} = e^{x^2/2}$

$\rightarrow y = \frac{1}{e^{x^2/2}}\int \frac{e^{x^2/2}}{x}\,dx$

$u = x^2/2 \quad | \quad = e^{-x^2/2}\int \frac{e^u}{x}\frac{du}{x} = e^{-x^2/2}\int \frac{e^u}{2u}\,du$

$du = x\,dx$

$dx = \frac{du}{x} \quad | \quad = e^{-\frac{x^2}{2}} * \frac{Ei(u)}{2} + c$

4) non-exact equations :-

three cases :-

* Case(1) :-

if $\mu = \mu(x) \rightarrow$ only

$\ln \mu - \ln c = -\int \frac{N_x - M_y}{N}\, dx$

* we will multiply μ * (function) → exact → solution

* Case (2) :-

$\mu = \mu(y)$ only

$\ln \mu = c\int \frac{N_x - M_y}{M}\, dy$

* Case (3) :-

$\mu = \mu(x, y)$

Partial differential equation

note:

the μ the integrating factor converts an non exact eq. to exact.

example :-

→ $(3xy + y^2) + (x^2 + xy)\frac{dy}{dx} = 0$

$\left.\begin{array}{l} \frac{\partial M}{\partial y} = 3x + 2y \\ \frac{\partial N}{\partial x} = 2x + y \end{array}\right]$ non exact

diff

$\ln\left(\frac{\mu}{c}\right) = -\int \frac{N_x - M_y}{N}\, dx$

$= -\int \frac{(2x + y) - (3x + 2y)}{x^2 + xy} = -\int \frac{-x + y}{x^2 + xy}\, dx$

$\frac{\ln \mu}{c} = \int \frac{x + y}{x(x+y)} = \ln|x|$

$\frac{\mu}{c} = x$

$\boxed{\mu = xc}$ (i.f)

$\mu * \left((3xy + y^2) + (x^2 + xy)\frac{dy}{dx} = 0\right)$

$(3x^2y + xy^2) + (x^3 + x^2y)\frac{dy}{dx} = 0$

$\underbrace{\frac{\partial M}{\partial x} = 3x^2 + 2xy, \quad \frac{\partial N}{\partial x} = 3x^2 + 2xy}_{\text{exact solution}}$

5] Bernoulli equation :

$dy/dx + P(x)y = Q(x)y^n$

$\mu(x) = e^{\int P_1(x)\, dx}$

$v = \frac{1}{\mu}\int \mu Q_1(x)\, dx$

$y = v^{-1}$

$P_1(x) = (1 - n)P(x)$

$Q_1(x) = (1 - n)Q(x)$

example :-

→ $y' + \frac{4}{x}y = x^3y^2$ | $P(x) = \frac{4}{x}$, $Q(x) = x^3$; $P_1(x) = \frac{4}{x} * -1$, $Q_1(x) = -x^3$

$\mu(x) = e^{\int -\frac{4}{x}dx} = e^{-4\ln|x|}$

$= x^{-4}$

$v = x^4\int (x^{-4} * -x^3)\, dx$

$= x^4\int -x^{-1}\, dx$

$= -x^4\ln(x) + c$

$y = \frac{-1}{x^4\ln(x)} + c_1$

6) Riccati equation :-

$$dy/dx = P(x)y^2 + Q(x)y + R(x)$$

Special Case :-

$P(x) = -1$

we assume $y = \frac{1}{u}\frac{du}{dx}$

So many steps we get to:-

$$u'' - Q(x)u' - R(x)u = 0 \quad ※$$

When we have constant coeff.

$u = e^{rx}$, $u' = re^{rx}$, $u'' = r^2e^{rx}$

$u = c_1e^{rx} + c_2e^{rx}$

$\frac{du}{dx} = \dots$

$y = \frac{du}{dx} / u$

example :-

$\rightarrow \quad dy/dx + y^2 = y + 6$

$P(x) = -1$, $Q(x) = 1$, $R(x) = 6$

assume $y = \frac{1}{u}\frac{du}{dx}$

$u = e^{rx}$

$u' = re^{rx}$

$u'' = r^2e^{rx}$

$$\boxed{y = \frac{-2c_1e^{-2x} + 3c_2e^{3x}}{c_1e^{-2x} + c_2e^{3x}}} \quad ※$$

$u'' - Q(x)u' - R(x)u = 0$

$r^2e^{rx} - re^{rx} - 6e^{rx} = 0$

$(r+2)(r-3)$

$r = -2$ | $u = c_1e^{-2x} + c_2e^{3x}$

$r = 3$ | $du/dx = -2c_1e^{-2x} + 3c_2e^{3x}$

7) linear equation coefficents

$$(ax + by + c)dx + (\alpha x + \beta y + \gamma)dy = 0$$

$* \ \ln x = -\int \frac{\alpha + \beta u}{\beta u^2 + (b+\alpha)u + a} du + c$

$* \ u = y/x$

example :-

$\rightarrow \quad (x + 2y)dx + x\,dy = 0$

$\rightarrow (x + 2y + 0)dx + (x + 0y + 0)dy = 0$

$\frac{\partial M}{\partial y} = 2$ (non-exact)

$\frac{\partial N}{\partial x} = 1$

$$\ln x = -\int \frac{1 + 0u}{0 + (2+1)u + 1} du$$

$$= -\int \frac{1}{3u+1} du$$

$\ln x = \frac{-\ln(3u+1)}{3} + c$

$3\ln x = -\ln(3u+1) + c_1$

$\ln(x^3(3u+1)) = c_1$

$x^3(3\frac{y}{x} + 1) = c_2$

$3yx^2 + x^3 = c_2$

$y = \frac{c_2 - x^3}{3x^2}$

8) first order of 2nd type

$$(dy/dx)^2 - \alpha\, dy/dx + \beta y = g(x)$$

- assume $P = dy/dx$

$$P^2 - \alpha P + \beta y = g(x)$$

$\uparrow dy/dx$

- $P(x) = \dfrac{\alpha \mp \sqrt{\alpha^2 - 4(\beta y - g(x))}}{2}$

- $u(x) =$ ما داخل الجذر

- $du/dx = \boxed{}$

(+) بكامل وبطلع بدلالة (y) (−) بكامل وبطلع بدلالة (y)

→ example :-

$$(dy/dx)^2 - 2\, dy/dx + y = x - 1$$

$P = dy/dx$

$P^2 - 2P + y = x - 1 \qquad \alpha = 2,\ \beta = 1$

$P = \dfrac{2 \mp \sqrt{(2)^2 - 4(y - x + 1)}}{2}$

$dy/dx = 1 \mp \sqrt{-y + x}$

$u = x - y$

$(du/\cancel{dx} = d\cancel{x} - dy) \times dx$

$\dfrac{du}{dx} = 1 - \dfrac{dy}{dx}$

$du/dx = 1 - (1 \mp \sqrt{u})$

$= \mp\sqrt{u}$

(+)

$du/dx = +\sqrt{u}$

$\int \dfrac{du}{\sqrt{u}} = \int dx$

$2\sqrt{u} = x + c$

$4u = (x + c)^2$

$u = \dfrac{(x + c)^2}{4}$

$x - y = (x + c)^2/4$

$y = x - (x + c)^2/4$

(−)

$du/dx = -\sqrt{u}$

$\int \dfrac{du}{\sqrt{u}} = \int -dx$

$2\sqrt{u} = -x + c$

$4u = (-x + c)^2$

$x - y = (-x + c)^2/4$

$y = x - (-x + c)^2/4$

9) Solution of system of homogenous linear (ODEs)

- $dy_1/dx = a_{11}y_1 + a_{12}y_2 + a_{13}y_3 + \cdots$
- $dy_2/dx = a_{21}y_1 + a_{22}y_2 + a_{23}y_3 + \cdots$
- $dy_3/dx = a_{31}y_1 + a_{32}y_2 + a_{33}y_3 + \cdots$

→ $dy/dx = Ay$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

* $y(x) = c_1 u_1 e^{\lambda_1 x} + c_2 u_2 e^{\lambda_2 x} + c_3 u_3 e^{\lambda_3 x} \ldots$

c_1, c_2, c_3 constants. / $\lambda_1, \lambda_2, \lambda_3$ eigen values

$$\underbrace{U_1 = \begin{bmatrix} U_{11} \\ V_{21} \\ U_{31} \end{bmatrix}, \quad U_2 = \begin{bmatrix} U_{21} \\ U_{22} \\ U_{23} \end{bmatrix}, \quad U_3 = \begin{bmatrix} U_{31} \\ U_{32} \\ U_{33} \end{bmatrix}}_{\text{eigen vectors}}$$

* eigen value and eigen vector

$AX = 0$, A ⇝ matrix of value - X ⇝ matrix of vectors

example :-

- $3x_1 + x_2 = 0$
- $2x_1 + 4x_2 = 0$

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Matrices :-

$$A = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & a_{11} & a_{12} \\ 2 & a_{21} & a_{22} \end{array} \xrightarrow{\text{Row}} \quad \downarrow \text{column} \quad \rightsquigarrow \underset{(R)}{2} \times \underset{(C)}{2}$$

$$A = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & a_{11} & a_{12} & a_{13} \\ 2 & a_{21} & a_{22} & a_{23} \\ 3 & a_{31} & a_{32} & a_{33} \end{array} \rightsquigarrow 3 \times 3$$

- Vectors :-

$$X = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rightsquigarrow 2 \times 1$$

$$X = \begin{bmatrix} 5 & 8 \end{bmatrix} \rightsquigarrow 1 \times 2$$

* matrix become vector if Row, column = 1
  - 1 Row and n Columns → Row vector
  - 1 column and n Rows → column vector

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$\Rightarrow A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$

$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$A = \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}$

$\Rightarrow A+B = \begin{bmatrix} 6 & 12 \\ 8 & 10 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 5 \\ 2 & 2 \end{bmatrix}$

- if $A(2\times2)$, $B(2\times1)$ → you cant find $(A+B)$
we must have same number of Rows and columns.

* multiplication :-

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} * \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}b_{11}+a_{12}b_{21}) & (a_{12}b_{12}+a_{12}b_{22}) \\ (a_{21}b_{11}+a_{22}b_{21}) & (a_{21}b_{12}+a_{22}b_{22}) \end{bmatrix}$$

ex :-

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} * \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} (1*5+3*7) & (1*6+3*8) \\ (2*5+4*7) & (2*6+4*8) \end{bmatrix}$$

$$= \begin{bmatrix} 26 & 30 \\ 38 & 44 \end{bmatrix}$$

- Matrix x vector

$[2\times\cancel{3}]\,[\cancel{3}\times1] = [2\times1]$

$$\underset{2\times3}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \begin{bmatrix} (a_{11}x_1+a_{12}x_2+a_{13}x_3) \\ (a_{21}x_1+a_{22}x_2+a_{23}x_3) \end{bmatrix}$$

→ Back to eigen value and eigen vector.

$AX = 0 \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow a_{11}x_1 + a_{12}x_2 = 0$

$a_{21}x_1 + a_{22}x_2 = 0$

$\sim AX = \lambda I X$ $\quad$ λ constant , $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ identity matrix

$AX - \lambda I X = 0$

$(A - \lambda I)X = 0$

$A - \lambda I = 0$

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} \boxed{a_{11}-\lambda} & a_{12} \\ a_{21} & \boxed{a_{22}-\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$AX = \lambda I X = 0$

$\lambda x_1 + 0x_2 = 0$

$0x_1 + \lambda x_2 = 0$

$I\lambda X = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

يتبع ←

$(a_{11}-\lambda)(a_{22}-\lambda) - (a_{12})(a_{21}) = 0$

$a_{11}a_{22} - a_{11}\lambda - a_{22}\lambda + \lambda^2 - a_{12}a_{21} = 0$

$a\lambda^2 - \underbrace{(a_{11}+a_{22})}_{b}\lambda + \underbrace{a_{11}a_{22} - a_{12}a_{21}}_{c} = 0$

$$\lambda = \frac{(a_{11}+a_{22}) \mp \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2a \to (1)}$$

- $\lambda_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$
- $\lambda_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

→ (eigen values)

example:-

→ $AX = 0$ $\quad A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ $\quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

let $\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$

$(1-\lambda)(3-\lambda) - 8 = 0$

$3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$

$\lambda^2 - 4\lambda - 5 = 0$

$(\lambda+1)(\lambda-5)$ $\quad \boxed{\lambda = -1,\ \lambda = 5}$ eigen value *

to get eigen vectors :-

- start with $\boxed{\lambda = -1}$

$$\begin{bmatrix} 1-(-1) & 2 \\ 4 & 3-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x_1 + 2x_2 = 0$

$4x_1 + 4x_2 = 0$

$x_1 = -x_2 \quad , \quad x_1 = -x_2$

assume

$x_1 = 1 \quad , \quad x_2 = -1$

$x_1 = 2 \quad , \quad x_2 = -2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad *$$

(eigen vectors) when $\lambda = -1$

when $\lambda = 5$:-

$$\begin{bmatrix} 1-(5) & 2 \\ 4 & 3-(5) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-4x_1 + 2x_2 = 0$ → $x_2 = 2x_1$

$4x_1 - 2x_2 = 0$ $\quad x_2 = 2x_1$

assume $x_1 = 1 \to x_2 = 2$, $x_1 = 5 \to x_2 = 10$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \beta \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad *$$

(eigenvector)

Solution of system of (ODEs)

① $dy_1/dx = a_{11}y_1 + a_{12}y_2$
$dy_2/dx = a_{21}y_1 + a_{22}y_2$ $\Big\} \rightarrow$ $dy/dx = ky$ (general form)

$\Rightarrow \int dy/y = \int k\,dx \rightarrow \ln y = kx + c \rightsquigarrow y = ce^{kx}$

② $dy_1/dx = \lambda y_1 \Rightarrow y_1 = Ae^{\lambda_1 x}$
$dy_2/dx = \lambda_2 y_2 \Rightarrow y_2 = Be^{\lambda_2 x}$

② in ①
$\lambda_1 Ae^{\lambda_1 x} = a_{11}Ae^{\lambda_1 x} + a_{12}Be^{\lambda_2 x}$
$\lambda_2 Be^{\lambda_2 x} = a_{21}Ae^{\lambda_1 x} + a_{22}Be^{\lambda_2 x}$

$\Rightarrow \lambda A = a_{11}A + a_{12}B$ because $e \neq 0$
$\lambda_2 B = a_{21}A + a_{22}B$ $\Big\} \rightarrow$ بنصف المعادلة وبنوخذ الثوابت $= 0$

let $\lambda_1 = \lambda_2 = \lambda \Rightarrow A\lambda = a_{11}A + a_{12}B$
$B\lambda = a_{21}A + a_{21}B$

$\rightarrow (a_{11} - \lambda)A + a_{12}B = 0$
$a_{21}A + (a_{22} - \lambda)B = 0$

$$\rightarrow \begin{bmatrix} a_{11}-\lambda & a_{21} \\ a_{21} & a_{22}-\lambda \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

let: $\begin{bmatrix} \ \end{bmatrix} = 0 \Rightarrow \lambda_1, \lambda_2$ Eigen values

substitute for each $\lambda_1, \lambda_2 \rightarrow$ gives A, B

$$y(x) = c_1 u_1 e^{\lambda_1 x} + c_2 u_2 e^{\lambda_2 x}$$

$u_1 = \alpha \begin{bmatrix} A \\ B \end{bmatrix}$, $u_2 = \beta \begin{bmatrix} A \\ B \end{bmatrix}$

example:-

solve:- $dy_1/dx = 6y_1 - 3y_2$
$dy_2/dx = 2y_1 + y_2$

$A = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}$, $A - \lambda I = \begin{bmatrix} 6-\lambda & -3 \\ 2 & 1-\lambda \end{bmatrix}$

$$\begin{bmatrix} 6-\lambda & -3 \\ 2 & 1-\lambda \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

let: $(6-\lambda)(1-\lambda) + 6 = 0$
$6 - 6\lambda - \lambda + \lambda^2 + 6 = 0$
$\lambda^2 - 7\lambda + 12 = 0$
$(\lambda - 3)(\lambda - 4)$
$\lambda = 3, \lambda = 4$

يتبع $\rightarrow$

when $\lambda = 4$: $\begin{bmatrix} 6-4 & -3 \\ 2 & 1-4 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2A - 3B = 0 \rightarrow B = 2/3 A$

$2A - 3B = 0 \rightarrow B = 2/3 A$

$\rightarrow U_1 = \begin{bmatrix} A \\ B \end{bmatrix} = \alpha \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

When $\lambda = 3$: $\begin{bmatrix} 6-3 & -3 \\ 2 & 1-3 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3A - 3B = 0 \rightarrow A = B$

$2A - 2B = 0 \rightarrow A = B$ $\rightarrow U_2 = \begin{bmatrix} A \\ B \end{bmatrix} = \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$y = C_1 \alpha \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4x} + C_2 \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3x}$$

$\rightarrow y_1 = 3 C_1 \alpha e^{4x} + C_2 \beta e^{3x}$
$= 3 C_{11} e^{4x} + C_{22} e^{3x}$

$\rightarrow y_2 = 2 C_1 \alpha e^{4x} + C_2 \beta e^{3x}$
$= 2 C_{11} e^{4x} + C_{22} e^{3x}$ *

*eigen values ① → real + distinct $\{\lambda_1, \lambda_2 \rightarrow \lambda_1 \neq \lambda_2\}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{\lambda_1 x} + C_2 \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} e^{\lambda_2 x}$$

② → Real + repeated $\{\lambda_1 = \lambda_2 = \lambda \text{ (real)}\}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} e^{\lambda x} + C_2 \left(x \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) e^{\lambda x}$$

③ → complex roots $\{\lambda_{1,2} = \alpha \pm \beta i\}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \left\{ C_1 \left(\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \cos\beta x - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \sin\beta x \right) + C_2 \left(\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \sin\beta x + \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \cos\beta x \right) \right\} e^{\alpha x}$$

example :- Repeated roots

$y_1' = 3y_1 + 13y_2$

$y_2' = 3y_2$

$$\begin{bmatrix} dy_1/dx \\ dy_2/dx \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 13 \\ 0 & 3 \end{bmatrix}}_{A} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

eigen values :- $A - \lambda I$

let $\begin{bmatrix} 3-\lambda & 13 \\ 0 & 3-\lambda \end{bmatrix} = 0 \Rightarrow (3-\lambda)^2 - 13 \times 0 = 0$

$(3-\lambda)^2 = 0 \rightarrow \lambda = 3, 3$

1st eigenvector. $\begin{bmatrix} 0 & 13 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$0U_1 + 13U_2 = 0 \Rightarrow U_2 = 0$

$U_1 = 1 \rightarrow U_2 = 0 \quad \rightarrow U = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$U_1 = 10 \rightarrow U_2 = 0$

$(A-3I)U = 0$

$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3x}$ ※ first sol.^{s}

• Solve $(A-3I)\omega = U$

$\rightarrow \begin{bmatrix} 3-3 & 13 \\ 0 & 3-3 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 13 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\rightarrow 0\omega_1 + 13\omega_2 = 1 \rightarrow \omega_2 = 1/13$

$\rightarrow \omega = \begin{bmatrix} 0 \\ 1/13 \end{bmatrix} = c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\rightarrow y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = (Ux + \omega)e^{3x} =$

$c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{3x}$

$\rightarrow y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3x} + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{3x}$

case ③ :-

example :-

$dy_1/dx = 2y_1 - 5y_2$
$dy_2/dx = y_1 + 2y_2 \Rightarrow \begin{bmatrix} dy_1/dx \\ dy_2/dx \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

. $A - \lambda I \rightarrow$ let $\begin{bmatrix} 2-\lambda & -5 \\ 1 & 2-\lambda \end{bmatrix} = 0$

$(2-\lambda)^2 + 5 = 0 \rightarrow 4 - 2\lambda - 2\lambda + \lambda^2 + 5 = 0$

$\lambda^2 - 4\lambda + 9 = 0 \rightarrow \lambda = 2 \mp i\sqrt{5}$ complex Root

• $\alpha = 2$, $\beta = \sqrt{5}$

① $\Rightarrow \begin{bmatrix} 2-(2+i\sqrt{5}) & -5 \\ 1 & 2-(2+i\sqrt{5}) \end{bmatrix} = \begin{bmatrix} -i\sqrt{5} & -5 \\ 1 & -i\sqrt{5} \end{bmatrix}$

$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\rightarrow -i\sqrt{5}U_1 - 5U_2 = 0 \rightarrow U_2 = -i\sqrt{5}/5\, U_1$

$\rightarrow U_1 - i\sqrt{5}U_2 = 0 \rightarrow U_2 = i\, 1/\sqrt{5}\, U_1$

$U = \begin{bmatrix} 1 \\ +1/\sqrt{5}\, i \end{bmatrix} = \begin{bmatrix} 1 + 0i \\ 0 + 1/\sqrt{5}\, i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{p} + \underbrace{\begin{bmatrix} 0 \\ 1/\sqrt{5} \end{bmatrix}}_{q} i$

$\rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = e^{2x} \Big\{ c_1 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos\sqrt{5}x - \begin{bmatrix} 0 \\ 1/\sqrt{5} \end{bmatrix} \sin\sqrt{5}x \right) +$

$c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin\sqrt{5}x - \begin{bmatrix} 0 \\ 1/\sqrt{5} \end{bmatrix} \cos\sqrt{5}x \right) \Big\}$

Topic 2.2

Applications of first-order ODE in Chemical Engineering

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Topic 2.2

Applications of first-order ODEs in Chemical Engineering

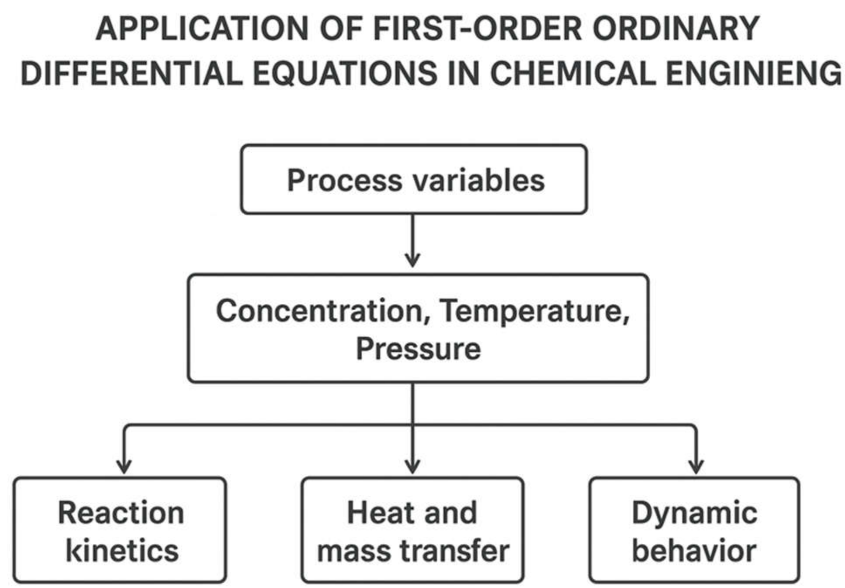
This topic covers the following applications of first-order ODEs in chemical engineering:

1. Fluid Mechanics and Mixing
2. Heat Transfer and Thermal Systems
3. Mass Transfer and Diffusion
4. Reaction Engineering
5. Process Dynamics and Control
6. Chemical Separation Processes
7. Environmental and Biochemical Systems

The next topic will be on the solution of second-order ODEs

Introduction

First-order ordinary differential equations are widely used in chemical engineering to describe how process variables such as concentration, temperature, and pressure change with time or position. They are essential for modeling reaction kinetics, heat and mass transfer, and dynamic behavior in reactors and process equipment. These equations help engineers predict system performance, optimize designs, and control industrial operations efficiently.

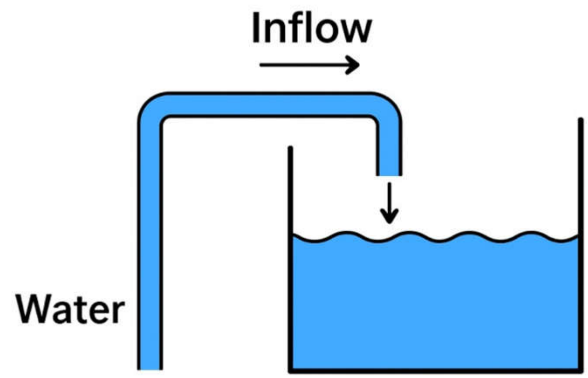


Fluid Mechanics and Mixing

Process	First-Order ODE	Description
Filling Tank (Constant Inflow)	$\frac{dh}{dt} = \frac{q_{in}}{A}$	Linear rise of liquid height.
Filling Tank (Inflow + Outflow)	$\frac{dh}{dt} = \frac{q_{in} - k\sqrt{h}}{A}$	Nonlinear ODE; level approaches steady-state.
Mixing Tank (Concentration Change)	$\frac{dC}{dt} = \frac{1}{\tau}(C_{in} - C)$	Solute concentration approaches inlet value exponentially.
Sedimentation Velocity	$\frac{dv}{dt} = g(1 - \frac{\rho_f}{\rho_p}) - \frac{3\mu v}{\rho_p d_p^2}$	Terminal velocity of particles.

Filling Tank with Constant Inflow

A tank is being filled with a constant volumetric flow rate, q_{in}(m³/s). The tank has a uniform cross-sectional area, A (m²). There is no outflow — all the fluid entering the tank accumulates inside. If $q_{in} = 0.002\ m^3/s$ and $A = 0.5\ m^2$, and the tank is initially empty, determine the level of water after 10 s.



Take the tank as the control volume. The mass inside the tank at time t is

$$m(t) = \rho V(t)$$

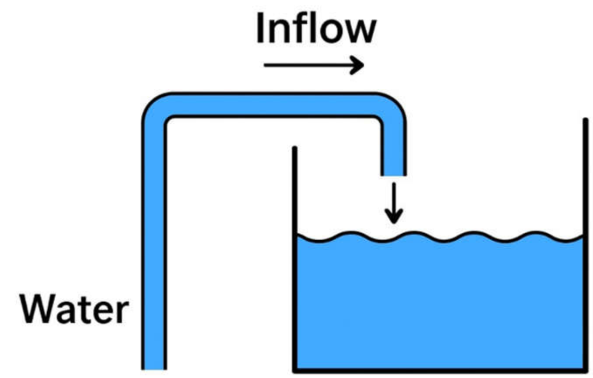
where

- m is mass (kg),
- ρ is fluid density (kg/m³) — assumed constant (incompressible),
- $V(t)$ is liquid volume in tank (m³).

Mass conservation (general) gives

$$\boxed{\frac{dm}{dt} = \dot{m}_{in}(t) - \dot{m}_{out}(t)}$$

where $\dot{m}_{in}$ and $\dot{m}_{out}$ are mass flow rates in and out (kg/s).



Relate mass flow rates to volumetric flow rates q (m^3/s):

$$\dot{m}_{in} = \rho\, q_{in}, \qquad \dot{m}_{out} = \rho\, q_{out}.$$

Substitute: $\Rightarrow$ $\rho \dfrac{dV}{dt} = \rho q_{in} - \rho q_{out}$ $\Rightarrow$ $\dfrac{dV}{dt} = q_{in}(t) - q_{out}(t)$

Assume uniform cross-sectional area A(m^2), with only water inflow. Then $V(t) = A\, h(t)$

$$A\frac{dh}{dt} = q_{in}(t) \quad \Rightarrow \quad \frac{dh}{dt} = \frac{q_{in}(t)}{A} \quad \Rightarrow \quad h(t) = h_0 + \frac{1}{A}\int_0^t q_{in}(\tau)\, d\tau$$

If q_{in} is constant (call it q_{in}):

$q_{in} = 0.002\ m^3/s$ and $A = 0.5\ m^2$, $h_0 = 0\ m$, determine the level of water after 10 s.

$$\frac{dh}{dt} = \frac{q_{in}}{A} \implies h(t) = h_0 + \frac{q_{in}}{A}\,t$$

Given: $q_{in} = 0.002\ \mathrm{m}^3/\mathrm{s},\ A = 0.5\ \mathrm{m}^2,\ h_0 = 0\ \mathrm{m},\ t = 10\ \mathrm{s}.$

$$h(t) = 0 + \left(\frac{0.002\ \mathrm{m}^3/\mathrm{s}}{0.5\ \mathrm{m}^2}\right) t \quad \Rightarrow \quad h(t) = 0.004\,t\ (\mathrm{m})$$

For a given time of 10 s, then the water level after this time becomes

$$h(10) = 0.004 \times 10 = 0.04\ \mathrm{m}$$

Heat Transfer and Thermal Systems

Process	First-Order ODE	Description
Newton's Law of Cooling	$\frac{dT}{dt} = -k(T - T_s)$	Object temperature approaches surroundings.
Heating a Tank (Energy Balance)	$\rho C_p V \frac{dT}{dt} = \dot{m} C_p (T_{in} - T) + Q$	Describes transient liquid temperature.
Heat Exchanger Start-Up	$\frac{dT_h}{dt} = \frac{UA}{\rho C_p V}(T_c - T_h)$	Hot fluid cools exponentially to steady-state.
Catalyst Particle Heating	$\frac{dT}{dt} = hA/(mC_p)(T_g - T)$	Lumped model of particle–gas heat transfer.

Newton's Law of Cooling - Basic Concept

A cup of hot coffee at 90°C is left to cool in a room at 25°C. After 5 minutes, the coffee temperature is 70°C. Find the coffee temperature after 10 minutes.

The rate of change of temperature of an object is proportional to the difference between its temperature and the surrounding (ambient) temperature:

For a closed system (no mass enters or leaves):

$$\frac{dE_{sys}}{dt} = \dot{Q} + \dot{W}$$

- E_{sys} = total energy of the system
- $\dot{Q}$ = rate of heat transfer into the system
- $\dot{W}$ = rate of work done on the system

For this case:

- No work ($W˙$=0W˙=0)
- The only energy exchange is heating loss from the coffee to the surroundings.

$$\Rightarrow \frac{dE_{sys}}{dt} = \dot{Q}$$

$$E_{sys} = mc_pT$$

Differentiate with respect to time:

$$\boxed{\frac{dE_{sys}}{dt} = \dot{Q}}$$

$$\frac{dE_{sys}}{dt} = mc_p \frac{dT}{dt}$$

Substitute into the energy balance: $\Rightarrow$ $$mc_p \frac{dT}{dt} = \dot{Q}$$

According to **Newton's law of cooling**:

$$\dot{Q} = -hA(T - T_a)$$

- h = convective heat transfer coefficient (W/m²·K)
- A = surface area exposed to air (m²)
- $T - T_a$ = temperature difference between coffee and air
- The negative sign indicates heat leaves the system.

Combine the Two Equations

$$mc_p \frac{dT}{dt} = -hA(T - T_a) \quad \Rightarrow \quad \frac{dT}{dt} = -\frac{hA}{mc_p}(T - T_a)$$

$$k = \frac{hA}{mc_p} \quad \Rightarrow \quad \boxed{\frac{dT}{dt} = -k(T - T_a)}$$

$$\boxed{\frac{dT}{dt} = -k(T - T_a)}$$

Integrate:

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

where T_0 is the initial temperature at $t = 0$.

$$T_0 = 90^\circ C \quad \text{(initial temperature)}$$

$$T_a = 25^\circ C \quad \text{(ambient temperature)}$$

$$T(5) = 70^\circ C \quad \text{(after 5 minutes)}$$

$$T(10) = ? \quad \text{(after 10 minutes)}$$

Substitute known values for $t = 5$:

$$70 = 25 + (90 - 25)e^{-5k}$$

$$\boxed{k = 0.0735 \text{ min}^{-1}}$$

Find Temperature at $t = 10$ min

$$T(10) = 25 + (90 - 25)e^{-0.0735(10)}$$

$$T(10) = 25 + 65e^{-0.735}$$

$$T(10) = 25 + 65(0.480)$$

$$T(10) = 56.2^\circ C$$

Mass Transfer and Diffusion

Process	First-Order ODE	Description
Gas Absorption (Film Theory)	$\frac{dC_s}{dt} = k(C_b - C_s)$	Surface concentration approaching bulk value.
Adsorption Kinetics (Pseudo-First-Order)	$\frac{dq}{dt} = k_1(q_e - q)$	Solute adsorbed onto solid with time.
Membrane Transport	$\frac{dC}{dx} = -\frac{J}{D}$	Steady-state diffusion through a membrane.
Drying Curve (Falling Rate Period)	$\frac{dX}{dt} = -k(X - X_{eq})$	Moisture content decays exponentially.

Perfume Dispersion in a Closed Classroom

A person sprays perfume in one corner of a closed classroom. Immediately after spraying, the air near that corner attains a perfume vapor concentration of $C_{Ao} = 2 \times 10^{-3}\,\frac{mol}{m^3}$. The perfume then begins to spread toward the opposite corner. Find the time of his perfume to reach the other corner (4m away from the first corner). $k \sim 0.001\,\mathrm{s}^{-1}$

$$\text{In} - \text{Out} + \cancel{\text{Generation}} - \cancel{\text{Consumption}} = \text{Accumulation}$$

$$\frac{dC_A}{dt} = k(C_{A0} - C_A)$$

C_A is the perfume concentration at the far corner.
C_{Ao} is the source concentration,
k is the **mass transfer coefficient**, which depends on distance and air mixing,

This is a standard linear ODE. Its solution is:

$$C_A(t) = C_{A0}\left(1 - e^{-kt}\right)$$

Let's assume the perfume is **detectable at** $C_A = C_{\text{detect}} = 10^{-6}\,\text{mol/m}^3$

$$C_A(t) = C_{A0}\left(1 - e^{-kt}\right)$$

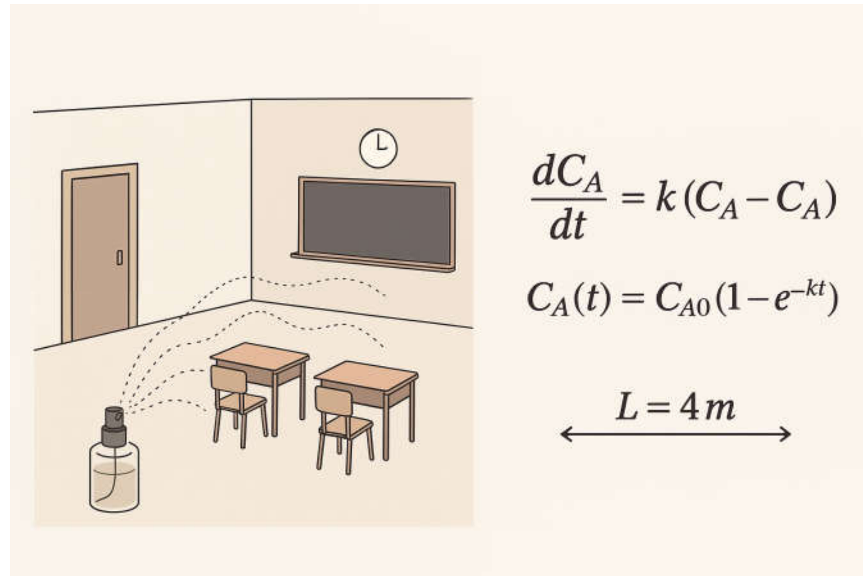
$$10^{-6} = 2 \times 10^{-3}\left(1 - e^{-kt}\right)$$

$$\frac{10^{-6}}{2 \times 10^{-3}} = 1 - e^{-kt}$$

$$5 \times 10^{-4} = 1 - e^{-kt} \implies e^{-kt} = 0.9995$$

$$-kt = \ln(0.9995) \implies t = -\frac{\ln(0.9995)}{k}$$

$$t = -\frac{-0.0005}{0.001} = 0.5\,\text{s}$$

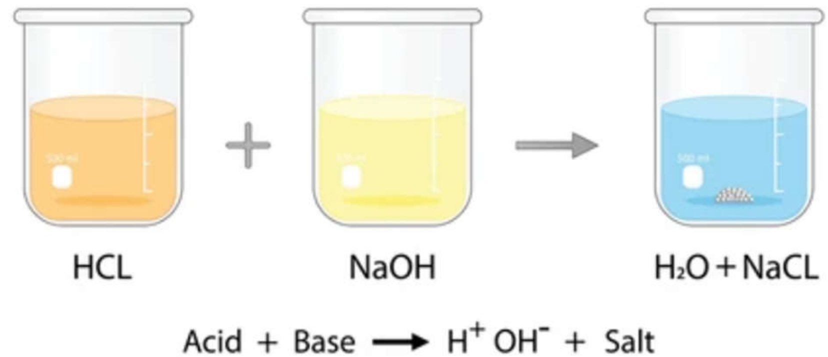


Reaction Engineering

Process	First-Order ODE	Description
Batch Reactor (1st-order)	$\frac{dC_A}{dt} = -kC_A$	Reactant concentration decays exponentially with time.
Continuous Stirred-Tank Reactor (CSTR)	$\frac{dC_A}{dt} = \frac{F}{V}(C_{A0} - C_A) - kC_A$	Describes transient reactor concentration before steady-state.
Plug Flow Reactor (PFR)	$\frac{dC_A}{dV} = -\frac{r_A}{F_A}$	Concentration along reactor volume.
Autocatalytic Reaction	$\frac{dC_A}{dt} = -kC_AC_B$	Can reduce to first order if C_B is constant.

Production of table salt

In a batch reactor, equal volumes of hydrochloric acid (HCl) and sodium hydroxide (NaOH) are mixed to produce sodium chloride. Both reactants have initial concentrations of 0.1 mol/L. The reaction is second order, and the rate constant: k = 1.0 L/(mol·s).
The neutralization reaction occurs according to:



$$\mathrm{HCl} + \mathrm{NaOH} \longrightarrow \mathrm{NaCl} + \mathrm{H_2O}$$

Determine the time required for [HCl] to decrease to 0.02 mol/L.

$$\frac{dC}{dt} = -k\,C^2 \quad \Rightarrow \quad C(t) = \frac{1}{k\,t + \frac{1}{C_0}} \quad \Rightarrow \quad t = \frac{1}{k}\left(\frac{1}{C} - \frac{1}{C_0}\right)$$

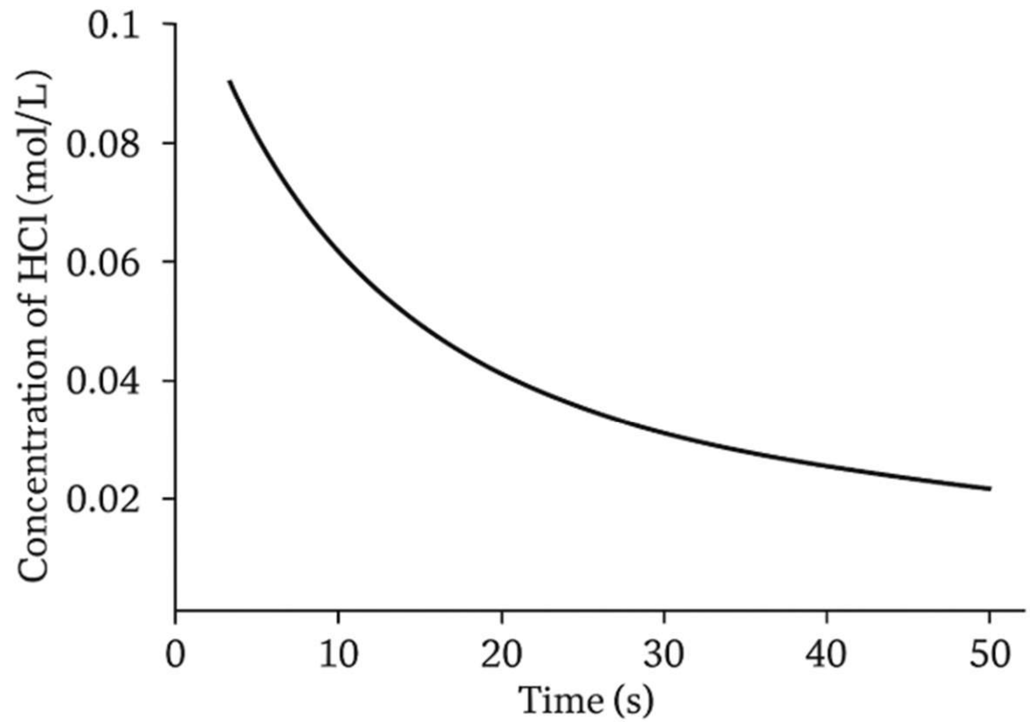
$$t = \frac{1}{k}\left(\frac{1}{C} - \frac{1}{C_0}\right)$$

- Initial concentration: $C_0 = 0.1\,\mathrm{mol/L}$
- Final concentration: $C = 0.02\,\mathrm{mol/L}$

$$t = \frac{1}{1}\left(\frac{1}{0.02} - \frac{1}{0.1}\right)$$

$$t = 50 - 10$$

$$t = 40\,\mathrm{s}$$

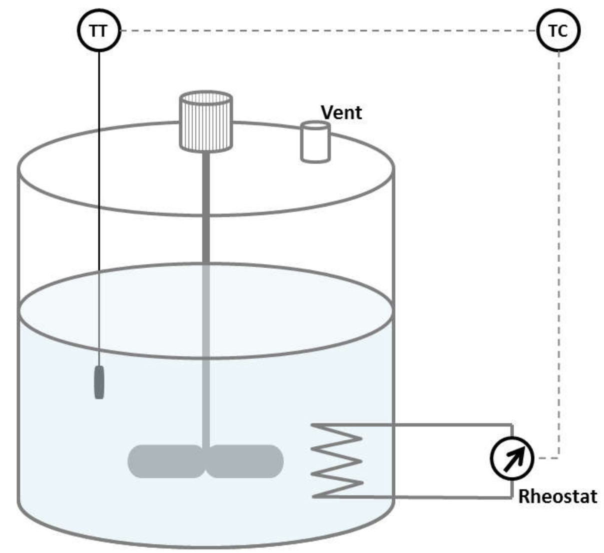


Process Dynamics and Control

Process	First-Order ODE	Description
Temperature Controller (First-Order Lag)	$\tau \frac{dT}{dt} + T = Ku(t)$	Dynamic response of a controlled process.
Pressure Response in a Gas Tank	$\frac{dP}{dt} = \frac{RT}{V}(F_{in} - F_{out})$	Pressure change with inflow/outflow.
Instrumentation (Thermocouple Response)	$\tau \frac{dT}{dt} + T = T_{\text{true}}$	Describes sensor lag.

First-order lag temperature controller

Process Dynamics and Control

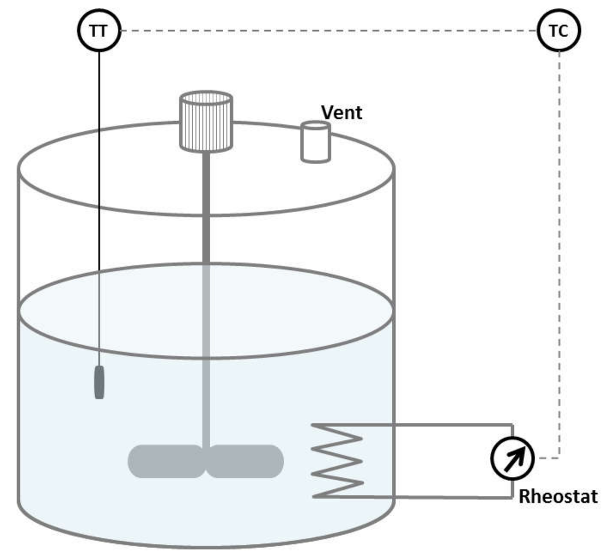


A small electrically heated, well-stirred tank is used to maintain the temperature of a process fluid. When the heater power is suddenly changed (a step input), the temperature of the liquid does not immediately reach its new steady-state value but follows a gradual rise described by a first-order lag behavior. Determine an expression for the variation of temperature with time.

Given Data

Parameter	Symbol	Value	Unit
Time constant	τ	5	s
System gain	K	20	°C/unit
Step input	$u(t)$	1 (for $t \geq 0$)	—
Initial temperature deviation	$\theta(0)$	0	°C

Process Dynamics and Control



Energy (first-law) balance

Accumulation = input − loss

$$mc_p\,\frac{dT}{dt} = Q_{\text{in}}(t) - Q_{\text{loss}}(t)$$

$$mc_p\,\frac{dT}{dt} = \alpha\,u(t) - hA\,[T(t) - T_{\text{env}}]$$

$$mc_p\,\frac{dT}{dt} + hA\,T(t) = \alpha\,u(t) + hA\,T_{\text{env}}$$

Write in deviation form by defining deviation from ambient:

$$\theta(t) \equiv T(t) - T_{\text{env}}$$

Since T_{env} is constant, $d\theta/dt = dT/dt$. Substitute: ⇨ $$mc_p\,\frac{d\theta}{dt} + hA\,\theta(t) = \alpha\,u(t)$$

Divide both sides by hA: ⇨ $$\frac{mc_p}{hA}\,\frac{d\theta}{dt} + \theta(t) = \frac{\alpha}{hA}\,u(t)$$

Let $\begin{cases} \tau \equiv \dfrac{mc_p}{hA} & \text{(time constant, s)} \\ K \equiv \dfrac{\alpha}{hA} & \text{(static gain, }^\circ\text{C per unit } u\text{)} \end{cases}$

$$\boxed{mc_p \frac{dT}{dt} + hA\,T(t) = \alpha\,u(t) + hA\,T_{\text{env}}}$$

$$\boxed{\frac{mc_p}{hA}\frac{d\theta}{dt} + \theta(t) = \frac{\alpha}{hA}u(t)}$$

$\tau = \frac{m\,C_p}{h\,A}$: Larger thermal mass ($mc_p$ (or smaller heat loss (hA) → **slower** response (larger τ).

$K = \frac{\alpha}{h\,A}$: More heater power per controller unit (α) or smaller loss (hA) → larger steady-state temperature change per unit u.

$$\frac{mc_p}{hA}\frac{d\theta}{dt} + \theta(t) = \frac{\alpha}{hA}u(t) \quad \Rightarrow \quad \tau\frac{d\theta}{dt} + \theta(t) = K\,u(t)$$

Divide by τ:

$$\frac{d\theta}{dt} + \frac{1}{\tau}\theta(t) = \frac{K}{\tau}u(t).$$

This is a first-order linear ODE: $\theta' + a\theta = b(t)$ with $a = \frac{1}{\tau}$ and $b(t) = \frac{K}{\tau}u(t)$.

$$\frac{d\theta}{dt} + \frac{1}{\tau}\theta(t) = \frac{K}{\tau}u(t)$$

$$\frac{d\theta}{dt} + P(t)\theta = Q(t) \qquad P(t) = \frac{1}{\tau}, \quad Q(t) = \frac{K}{\tau}u(t)$$

the integrating factor $\mu(t)$

$$\mu(t) = e^{\int P(t)dt} = e^{\int \frac{1}{\tau}dt} = e^{t/\tau}$$

$$\theta = \frac{1}{\mu}\int (\mu Q)\,dx \quad \Rightarrow \quad \theta(t) = e^{-t/\tau}\left[\int \frac{K}{\tau}u(t)e^{t/\tau}dt + C\right]$$

step input $u(t) = U_0$ $\Rightarrow$

$$\theta(t) = e^{-t/\tau}\left[\int_0^t \frac{K}{\tau}U_0 e^{t'/\tau}dt' + \theta(0)\right]$$

$$\theta(t) = KU_0\left(1 - e^{-t/\tau}\right) + \theta(0)e^{-t/\tau}$$

$K = 20\ {}^oC/unit$
$\tau = 5\ s$
$\theta(0) = 0$
$U_0 = 1$

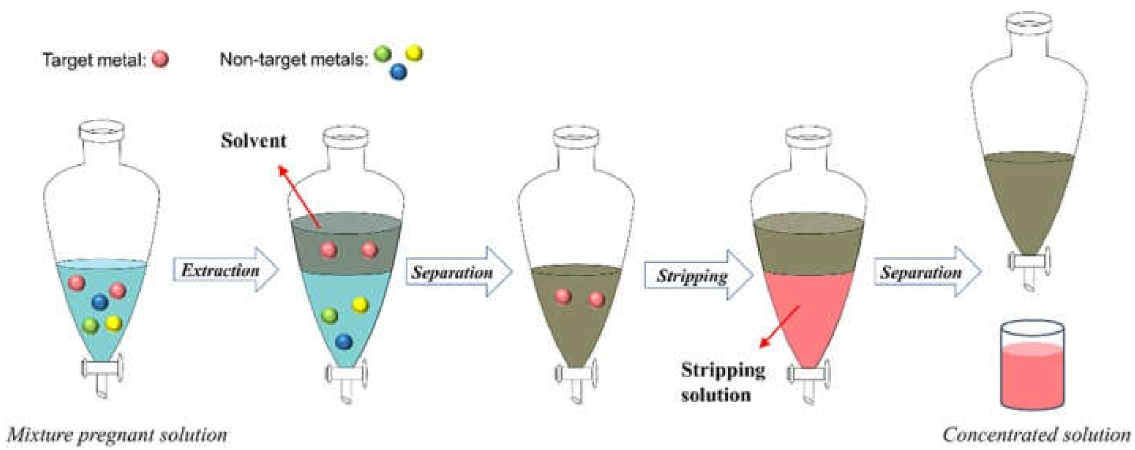
$$\Rightarrow \boxed{\theta(t) = 20\left(1 - e^{-t/5}\right)} \Rightarrow \boxed{T(t) = T_{env} + 20\left(1 - e^{-t/5}\right)}$$

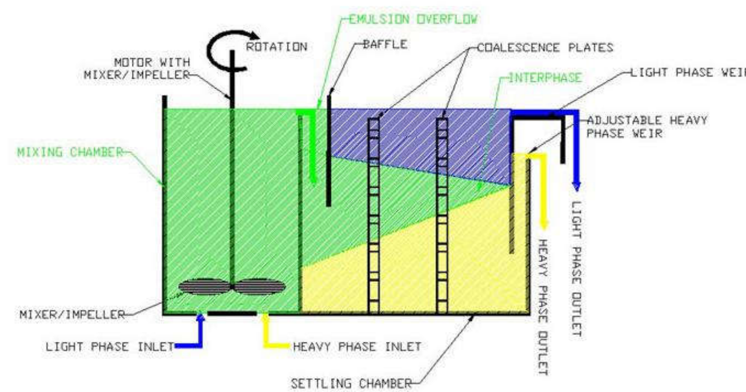
Separation Processes

Process	First-Order ODE	Description
Distillation (Dynamic Component Balance)	$\frac{dx_D}{dt} = \frac{1}{\tau}(x_{D,eq} - x_D)$	Top product composition dynamics.
Liquid–Liquid Extraction (Single Stage)	$\frac{dy}{dt} = k(y^* - y)$	Solute transfer toward equilibrium.
Gas–Liquid Absorber Dynamics	$\frac{dy}{dz} = K_y a(y^* - y)$	Change in gas concentration along height z.

Single-stage Extraction Unit

A single-stage liquid–liquid extraction is performed in a stirred tank. The feed contains a solute at a concentration of $C_F = 0.1\ mol/L$. The solvent is initially free of solute. The solute transfer rate into the solvent is proportional to the difference between the equilibrium concentration $C^* = 0.3\ mol/L$ and the current solute concentration in the solvent $C(t)$:





Single-stage Extraction Unit

General mass balance for the solute in the solvent phase: For a well-mixed solvent tank in a single-stage batch extraction:

$$\text{Accumulation} = \text{Inflow of solute} - \text{Outflow of solute} + \text{Generation of solute}$$

the **accumulation term** is:

m_{solute} = mass of solute in solvent

r_{transfer} = rate of mass transfer from feed to solvent

$$\frac{d(m_{\text{solute}})}{dt} = r_{\text{transfer}}$$

If the solvent has **volume** V (assumed constant), then:

$$m_{\text{solute}} = C(t)\,V$$

$$\frac{dm_{\text{solute}}}{dt} = V\frac{dC}{dt}$$

In liquid–liquid extraction, **mass transfer is proportional to the driving force**:

$$r_{\text{transfer}} = k(C^* - C(t))\,V$$

$$\frac{d(m_{\text{solute}})}{dt} = r_{\text{transfer}} \qquad \frac{dm_{\text{solute}}}{dt} = V\frac{dC}{dt} \qquad r_{\text{transfer}} = k(C^* - C(t))\,V$$

$C(t)$ = solute concentration in solvent at time t

C^* = equilibrium solute concentration in solvent if the phases were at equilibrium

k = mass transfer coefficient

$$V\frac{dC}{dt} = k(C^* - C)V \quad \Rightarrow \quad \frac{dC}{dt} = k(C^* - C)$$

General solution

$$C(t) = C^* + (C_0 - C^*)e^{-kt}$$

Substitute values: $C^* = 0.3$, $C_0 = 0$, $k = 0.2$:

$$\begin{aligned} C(t) &= 0.3 + (0 - 0.3)e^{-0.2t} \\ &= 0.3(1 - e^{-0.2t}) \end{aligned}$$

$$C(5) = 0.3(1 - e^{-0.2\cdot 5}) = 0.3(1 - e^{-1}) \quad \Rightarrow \quad \boxed{C(5) \approx 0.19\ \mathrm{mol/L}}$$

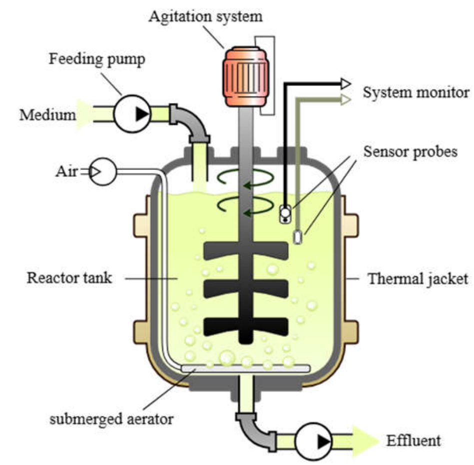
Environmental and Biochemical Systems

Process	First-Order ODE	Description
Biodegradation (1st-order decay)	$\frac{dC}{dt} = -kC$	Pollutant concentration in wastewater.
Microbial Growth (Log phase)	$\frac{dX}{dt} = \mu X$	Exponential biomass growth.
Atmospheric Contaminant Removal	$\frac{dC}{dt} = -k(C - C_{eq})$	Pollutant approaches equilibrium with surroundings.

Biomass growth in a bioreactor

Environmental and Biochemical Systems

A batch bioreactor contains a microbial culture with an initial biomass concentration of $X_0 = 0.2\ g/L$. The microorganisms grow in a nutrient-rich medium under ideal conditions. Assume the growth is exponential and limited only by time (nutrient excess). The specific growth rate is $\mu = 0.3\ h^{-1}$. Determine the biomass concentration $X(t)$at any time $t$and find the concentration after 6 hours.



Start from the fundamental mass balance: For a well-mixed batch reactor:

$$\text{Accumulation} = \text{Generation} - \text{Consumption} + \text{Inflow} - \text{Outflow}$$

Batch reactor → no inflow or outflow

No consumption of biomass except growth → Consumption = 0

$$\frac{d(\text{Biomass mass})}{dt} = \text{Rate of biomass generation}$$

Let V = reactor volume (constant), $X(t)$ = biomass concentration:

$$m_X = XV \quad \Rightarrow \quad \frac{dm_X}{dt} = V\frac{dX}{dt}$$

Biomass grows proportionally to its current concentration:

$$\text{Rate of biomass generation} = \mu XV$$

Where μ is the **specific growth rate**.

$$V\frac{dX}{dt} = \mu XV \quad \Rightarrow \quad \frac{dX}{dt} = \mu X$$

Separate variables: $\frac{dX}{X} = \mu dt$ ⇨ $\ln X = \mu t + C$ ⇨ $X(t) = X_0 e^{\mu t}$, $X(0) = X_0$

$X(t) = 0.2\, e^{0.3t} \quad \mathrm{g/L}$ ⇨ $X(6) = 0.2\, e^{0.3\cdot 6} = 0.2\, e^{1.8} \approx 1.21\ \mathrm{g/L}$

End of Topic 2.2

Chapter(2)

Solution of 2nd order ODEs

$$y'' + P(x)y' + q(x)y = g(x)$$

method (1):

$$y'' + a_1 y' + a_0 y = g(x)$$

constant coeff. 2nd order ODEs.

$\Rightarrow y(x) = y_c(x) + y_p(x)$

• $y_c(x)$?! $\Rightarrow y'' + a_1 y' + a_0 y = 0$

assume

$$y_c = e^{rx}, \quad y'_c = re^{rx}, \quad y_c'' = r^2 e^{rx}$$

$\Rightarrow r^2 e^{rx} + a_1 r e^{rx} + a_0 e^{rx} = 0$

$e^{rx}(r^2 + a_1 r + a_0) = 0$

$r^2 + a_1 r + a_0 = 0 \longrightarrow r_{1,2} = \dfrac{-a \mp \sqrt{a_1^2 - 4a_0}}{2}$

$r_1 = \dfrac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2}, \quad r_2 = \dfrac{-a - \sqrt{a_1^2 - 4a_0}}{2}$

$\Rightarrow y_c(x) = Ae^{r_1 x} + Be^{r_2 x}$

case (1): $r_1 \neq r_2$ (real roots)

$y_c(x) = Ae^{r_1 x} + Be^{r_2 x}$

case (2): $r_1 = r_2 = r$ (real roots)

$y_c(x) = (A + Bx)e^{rx}$

case(3): $r_1 \neq r_2$ (complex) $\quad r_1 = \alpha + \beta i, \; r_2 = \alpha - \beta i$

$y_c(x) = (A\cos\beta x + B\sin\beta x)\, e^{\alpha x}$

$\cdot\, y_p(x)$?! $\rightarrow y'' + a_1 y' + a_0 y = g(x)$
↳ (Particular solu.)

- if $g(x) = c \Rightarrow y_p(x) = \frac{c}{a_0}$
- if $g(x) = cx \Rightarrow y_p(x) = \frac{c}{a_0}\left[x - \frac{a_1}{a_0}\right]$
- if $g(x) = cx^2 \Rightarrow y_p(x) = \frac{c}{a_0}\left[x^2 - \frac{2a_1}{a_0} + \frac{2(a_1^2 - a_0)}{a_0^2}\right]$

---

example :: Solve $y'' + 3y' + 2y = 1$

$a_1 = 3,\ a_0 = 2,\ g(x) = 1$

- $r_1 = \frac{-3 \sqrt{9-4(2)}}{2}$ • $r_2 = \frac{-3 - \sqrt{9-4(2)}}{2}$

$r_1 = -1$ $r_2 = -2$

$r_1 \neq r_2$
↓
(real roots)

- $y_c(x) = Ae^{-x} + Be^{-2x}$
- $y_p(x) = 1/a_0 = 1/2$

$\Rightarrow y(x) = y_c(x) + y_p(x) = Ae^{-x} + Be^{-2x} + \frac{1}{2}$

- if $g(x) = 6x \rightsquigarrow y_p(x) = 6/2\,[x - 3/2]$
$= 3\,(x - 3/2)$

---

example :: Solve $y'' + y' + y = 5x^2$

$a_1 = 1,\ a_0 = 1,\ g(x) = 5x^2$

$r = \frac{-1 \mp \sqrt{1-4}}{2} = \frac{-1 \mp \sqrt{-3}}{2} = -1/2 \mp \frac{\sqrt{3}}{2} i$

$r_1 = -1/2 + \frac{\sqrt{3}}{2} i$, $r_2 = -1/2 - \frac{\sqrt{3}}{2} i$

$\alpha = -1/2$, $\beta = \sqrt{3}/2$

. $y_c(x) = (A\cos\sqrt{3}/2 + B\sin\sqrt{3}/2\,x)\, e^{-\frac{1}{2}x}$

. $y_p(x) = \frac{5}{1}\left[x^2 - 2 + \frac{2\ (1-1)}{1^2}\right]$
$= 5x^2 - 10$

. $y(x) = (A\cos\frac{\sqrt{3}}{2}x + B\sin\sqrt{3}/2\,x)\, e^{-\frac{1}{2}x} + 5x^2 - 10$

note::

* if $g(x) = g_1 + g_2 + \dots$, we should find $y_p(x)$ for every (g).

method (2) $y'' = f(x, y')$

$\Rightarrow y'' + P(x)y' = g(x) \rightsquigarrow y'' + 2xy' = x^2$

assume $P = y' \longrightarrow P' = dP/dx = y''$

example:

- $y'' + 2xy' = x$
- $\frac{dp}{dx} + 2xP = x$ * first order ODEs solved by intigrating factor

$\mu = e^{\int 2x\,dx} = e^{x^2}$

$y = e^{-x^2} \int e^{x^2} \cdot x \, dx$ * $u = x^2 \to du = 2x\,dx$

$y = e^{-u} \int e^{u} \cdot x \frac{du}{2x}$

$= 1/2\, e^{-x^2} \int e^u \, du = \frac{1}{2} e^{-x^2} (e^{x^2} + c)$

$= \frac{1}{2} + \frac{1}{2} c e^{-x^2} = \frac{1}{2} + Ae^{-x^2}$ * $(A = \frac{1}{2} c)$

$P = \frac{1}{2} + Ae^{-x^2} = dy/dx$

$\int dy = \int (\frac{1}{2} + Ae^{-x^2})\, dx$

$y = \frac{1}{2} x + A \int e^{-x^2} dx + B$

↳ will be solved later

---

method (3):

$y'' + P(x)y' + q(x)y = g(x)$

$y'' = f(x, y, y')$

assume x does not exist $\to y'' = f(y, y')$

example:

$1 + yy'' + y'^2 = 0$

$y'' + 2y' - y = 0$

$\to$ Solu:

$y' = P \;,\; y'' = P' = dP/dx = dP/dy \cdot dy/dx$

$= dP/dy \cdot y' = d/dy \cdot P$

$\to y'' = P' = P\, dP/dy$

example :: $1 + yy'' + y'^2 = 0$

$\rightarrow 1 + y\,p\,dp/dy + p^2 = 0$

$y\,p\,dp/dy = -(1+p^2)$

$$\frac{p\,dp}{(1+p^2)} = -\frac{dy}{y}$$

$$\frac{1}{2}\ln(1+p^2) = -\ln(y) + c$$

$$c = \frac{1}{2}\ln(1+p^2) + \ln(y) \quad *2$$

$$c_1 = \ln(1+p^2) + \ln(y^2)$$

$$c_1 = \ln[(1+p^2)\cdot y^2] \leftarrow \exp$$

$$(1+p^2)y^2 = c_2 = \alpha^2 \text{ (constant)}$$

$$1 + p^2 = \frac{\alpha^2}{y^2} = \left(\frac{\alpha}{y}\right)^2$$

$$\rightarrow p^2 = \left(\frac{\alpha}{y}\right)^2 - 1 \rightarrow p = \sqrt{\left(\frac{\alpha}{y}\right)^2 - 1} = y' = \frac{dy}{dx}$$

$$\frac{dy}{\sqrt{\left(\frac{\alpha}{y}\right)^2 - 1}} = dx \rightarrow (x+b)^2 + y^2 = \alpha^2 \ \#$$

---

method (4) ::

$$xy'' = f(y', y/x)$$

example ::

$$xy'' + y'^2 - (y/x)^2 = 0$$

assume $r = y/x \rightsquigarrow y = xr$

$$y' = xr' + r$$

$$y'' = xr'' + r' + r' = xr'' + 2r'$$

$$x[xr'' + 2r'] + [xr' + r]^2 - r^2 = 0$$

$$[x^2 r'' + 2xr'] + [x^2 r'^2 + 2xr'r + r^2] - r^2 = 0$$

$$x^2 r'' + 2xr' + 2xr'r + x^2 r'^2 = 0 \text{ - - - - - } \boxed{\text{I}}$$

assume $x = e^t \rightarrow t = \ln x$

$$r' = \frac{dr}{dx} = \frac{dr}{dt}\cdot\frac{dt}{dx} = \frac{dr}{dt}\left(\frac{1}{x}\right)$$

$$r'' = \frac{d^2 r}{dx^2} = \left[\frac{d^2 r}{dt^2} - \frac{dr}{dt}\right]\left(\frac{1}{x}\right)^2$$

(take it as it is)

$$\to x^2\left[\frac{d^2 r}{dt^2}-\frac{dr}{dt}\right]\left(\frac{1}{x^2}\right)+2x\left(\frac{dr}{dt}\right)\left(\frac{1}{x}\right)+$$

$$2xr\left(\frac{dr}{dt}\right)\left(\frac{1}{x}\right)+x^2\left(\frac{dr}{dt}\right)^2\left(\frac{1}{x^2}\right)=0$$

$$\to \frac{d^2 r}{dt^2}-\frac{dr}{dt}+2\frac{dr}{dt}+2r\frac{dr}{dt}+\left(\frac{dr}{dt}\right)^2=0$$

(assume) $P=\frac{dr}{dt}$

$$\frac{dr^2}{dt^2}=\frac{dP}{dt}=\frac{dP}{dr}\cdot\frac{dr}{dt}=\frac{dP}{dr}\cdot P$$

$$\to P\frac{dP}{dr}-P+2P+2rP+P^2=0$$

$$P\frac{dP}{dr}+P+2rP+P^2=0$$

$$P\frac{dP}{dr}+P(1+2r)+P^2=0 \quad (\div P)$$

$\frac{dP}{dr}+P=-(1+2r)$ ⇝ ODE solved by intigrating factor

توقف الفهم

$$\Rightarrow \quad \mu=e^{\int dr}=e^{r}$$

$$P=\frac{1}{\mu}\int \mu\, Q(r)\, dr$$

$$=e^{-r}\int e^{r}(1+2r)\,dr \longrightarrow \text{by parts}$$

$$P=1-2r+ce^{-r} \;\#$$

$$\Rightarrow \quad P=\frac{dr}{dt}=x\frac{dr}{dx}=\frac{x\,d(y/x)}{dx}$$

$$\frac{dy}{dx}-\frac{y}{x}=1-2\left(\frac{y}{x}\right)+c_2e^{(-y/x)}$$ ⇝ take it as it is

* Solution of 2nd ODE's :-

- series solution of ODE's - Frobeneous method

$$P_2(x)\,y'' + P_1(x)\,y' + P_0(x)\,y = 0$$

→ let $y = x^c \sum_{n=0}^{\infty} A\, x^{np} = x^c\left[A_0 x^0 + A_1 x^p + A_2 x^{2p} \dots\right]$

assume $y = x^c \rightarrow y' = c\,x^{c-1} \rightarrow y'' = c(c-1)\,x^{c-2}$

$\rightarrow P_2\, c(c-1)\, x^{c-2} + P_1\, c\, x^{c-1} + P_0 x^c = 0$

↳ $f(c)\, x^r + g(c)\, x^{r+p} = 0$

this term approches zero faster than the other term. So we can neglect it. that gives

$f(c)\, x^r = 0$, $x^r \neq 0$ so $f(c) = 0$

↳ from the difference between x^r, x^{r+p}, we defined p

↳ and c can determin from $f(c) = 0$. → c_1, c_2 .

$$\rightarrow A_{k+1} = (-1)^{k+1} A_0 \frac{g(c_1)\, g(c_1+p)\, g(c_1+2p)\dots}{f(c_1+p)\, f(c_1+2p)\, f(c_1+3p)\dots}$$

find A for c_1 and A for c_2

* these are three cases based on c values

---

case (1) :: c_1, c_2 are real and do not differ by Integer

↳ $c_1 - c_2$ should be fraction . ex. $c_1 = 0$, $c_2 = 1/2$

$c_1 = 3$, $c_2 = 3.4$ / $c_1 = 1$, $c_2 = 2$ ⤳ not valid in this case

Solu. $y = A y_1 + B y_2$

(A_{k+1}) from c_1 (B_{k+1}) from c_2

$y_1 = x^{c_1} + A_1 x^{c_1+p} + A_2 x^{c_1+2p} + \dots$

$y_2 = x^{c_2} + B_1 x^{c_2+p} + B_2 x^{c_2+2p} + \dots$

---

example :-

$2xy'' + y' + y = 0 \leftrightarrow P_2 y'' + P_1 y' + P_0 y = 0$

$P_2 = 2x$, $P_1 = 1$, $P_0 = 1$

$y = x^c$, $y' = c\,x^{c-1}$, $y'' = c(c-1)\, x^{c-2}$

$\rightarrow 2xc\,(c-1)\, x^{c-2} + c\,x^{c-1} + x^c = 0$

$= 2c\,(c-1)\, x^{c-1} + c\,x^{c-1} + x^c = 0$

$[2c(c-1) + c]\, x^{c-1} + x^c = 0$

$f(c)\,x^{c-1} + g(c)\,x^{c} = 0$

$\cdot\ f(c) = 2c^2 - 2c + c = 0$

$2c^2 - c = 0$

$c\,(2c-1) = 0$

$c = 0\ ,\ c = .5$

$g(c) = 1 \rightsquigarrow$ note :: value of $g(c)$ always should be (1)

$p = c - (c-1) = 1$

$\rightarrow$ for first solu. $c_1 = 0$

$y_1 = x^{c} + A_1\,x^{c+p} + A_2\,x^{c+2p} + \dots$

$= x^{0} + A_1\,x^{0+1} + A_2\,x^{0+2(1)} \dots$

$= 1 + A_1 x + A_2 x^2 + \dots$

$\rightarrow k=0 \rightarrow A_1 = (-1)^1 A_0\,\dfrac{g(c_1)}{f(c_1+p)}$

we take first term only $(f(c_1+p))(g(c_1))$

$= -A_0\,\dfrac{1}{2[0+1]^2 - [0+1]}$

$= -A_0$

$\cdot\ f(c) = 2c^2 - c\ ,\ f(c_1+p) = 2[c_1+p]^2 - [c_1+p]$

$\rightarrow k=1 \rightarrow A_2 = (-1)^2 A_0\,\dfrac{g(c_1)\,g(c_1+p)}{f(c_1+p)\,f(c_1+2p)}$

$= A_0\,\dfrac{1*1}{1*6} = \dfrac{A_0}{6}$

$\longrightarrow k=2 \rightarrow A_3 = (-1)^3 A_0\,\dfrac{g(c_1)\,g(c_1+p)\,g(c_1+2p)}{f(c_1+p)\,f(c_1+2p)\,f(c_1+3p)}$

$= -A_0\,\dfrac{1*1*1}{1*6*15} = \dfrac{-A_0}{90}$

$\rightarrow y_1 = A_0 x^{c} + A_1 x^{c_1+p} + A_2 x^{c_1+2p} + A_3 x^{c_1+3p}$

$= A_0 x^0 + (-A_0)\,x^1 + \dfrac{A_0}{6}x^2 + \dfrac{(-A_0)}{90}x^3$

$= A_0 - A_0 x + \dfrac{A_0}{6}x^2 - \dfrac{A_0}{90}x^3$

$y_1 = A_0\left[1 - x + \dfrac{1}{6}x^2 - \dfrac{1}{90}x^3\right]$

$$\rightarrow B_{k+1} = (-1)^{k+1} B_0 \frac{g(c_2+p)\, g(c_2+2p)\ldots}{f(c_2+p)\, f(c_2+2p)\, f(c_2+3p)\ldots}$$

$$k=0 \rightarrow B_1 = (-1)^1 B_0 \frac{1}{(\frac{1}{2}+1)(2(\frac{3}{2})-1)}$$

$$= -B_0 \frac{1}{3} = \frac{-B_0}{3}$$

$$K=2 \rightarrow B_3 = (-1)^3 B_0 \frac{1}{(3)(10)(\frac{1}{2}+3)(2(\frac{7}{2})-1)}$$

$$= \frac{-B_0}{(3)(10)(21)} = \frac{-B_0}{630}$$

$$\rightarrow y_2 = B_0 x^{\frac{1}{2}} \left[1 - \frac{1}{3}x + \frac{1}{30}x^2 - \frac{1}{630}x^3 \right]$$

$$\rightarrow y = y_1 + y_2$$

Case (2): Both Roots are equal $c_1 = c_2 = c$

Solu :- $y = \gamma y_1(x, c) + \delta \left(\frac{\partial y_1}{\partial c}\right)_{c=c_1}$

$$y_1 = x^r \sum a_n x^n$$

take it as it is

$$y_2 = y_1(x) \ln x + x^r \sum b_n x^n$$

example :- $xy'' + y' + xy = 0$ $P_2 = x$, $P_1 = 1$, $P_0 = x$

$y = x^c$, $y' = c x^{c-1}$, $y'' = c(c-1) x^{c-2}$

$$\rightarrow x c(c-1) x^{c-2} + c x^{c-1} + x \cdot x^c = 0$$

$$(c^2 - c) x^{c-1} + c x^{c-1} + x^{c+1} = 0$$

$$c^2 x^{c-1} + (1) x^{c+1} = 0$$

$f(c) = c^2$ $g(c) = 1$ $P = (c+1) - (c-1) = 2$

$\rightarrow c_1 = c_2 = 0$ repeated roots

$$\rightarrow y_1 = A_0 x^r + A_1 x^{r+p} + A_2 x^{r+2p} + A_3 x^{r+3p}$$

$$k=0 \rightarrow A_1 = (-1)^1 A_0 \frac{1}{(c+2)^2} = -A_0/4$$

$$K=1 \rightarrow A_2 = (-1)^2 A_0 \frac{(1)(1)}{(c+2)^2 (c+4)^2} = A_0/64$$

$$K=2 \rightarrow A_3 = (-1)^3 A_0 \frac{(1)}{(c+2)^2 (c+4)(c+6)^2} = \frac{-A_0}{(4)(16)(36)} = \frac{-A_0}{2304}$$

$$\rightarrow y_1 = A_0 x^c + A_1 x^{c+2} + A_2 x^{c+4}$$

$$= A_0 x^c - \frac{A_0}{(c+2)^2} x^{c+2} + \frac{A_0}{(c+2)^2(c+4)^2} x^{c+4} + \dots$$

$$= A_0 x^c \left[1 - \frac{x^2}{(c+2)^2} + \frac{x^4}{(c+2)^2(c+4)^2} + \dots \right]$$

$$y_1(c=0) = A_0 \left[1 - \frac{x^2}{4} + \frac{x^4}{64} + \dots \right]$$

$$\rightarrow y_2 = \frac{dy_1}{dc} = \frac{d}{dc}\left[A_0 x^c \left[1 - \frac{x^2}{(c+2)^2} + \frac{x^4}{(c+2)^2(c+4)^2} \right] \right]$$

$$= y_1 \ln x + x^c \left[1 + \frac{2x^2}{(c+1)^2} - \frac{2x^4}{(c+2)^2(c+4)^2} \right]$$

$$\rightarrow P_2(x) y'' + P_1(x) y' + P_0(x) y = 0$$

$y = x^c \rightarrow f(c) = x^r + g(c)\, x^{r+p} = 0$

$f(c) = 0 \longrightarrow$ roots $\rightarrow$ case 1 $c_1 \neq c_2$ (differ by fraction)

$\hookrightarrow$ case 2 $c_1 = c_2$

---

case(3) :: $c_1 \neq c_2$ differ by integer

$$y = A y_1(x, c_2) + B \frac{d}{dc}\left[(c - c_1) y(x, c) \right]_{c=c_2}$$

* c_1 = Smallest root , * c_2 = largest root

• $A_{k+1} = (-1)^{k+1} A_0 \dfrac{g(c)\, g(c+p)\, g(c+2p)}{f(c+p) f(c+2p) f(c+3p)}$

(Same for B)

---

example:: $xy'' + y = 0$

$P_2 = x$, $P_1 = 0$, $P_0 = 1$

$\rightarrow y = A y_1(x) \ln x + \sum_{0}^{\infty} b_n x^{n+r}$ ⤳ largest root

$\rightarrow y = x^c$, $y' = c x^{c-1}$, $y'' = c(c-1) x^{c-2}$

$x c(c-1) x^{c-2} + x^c = 0$

$c(c-1) x^{c-1} + x^c = 0$

$\rightarrow f(c) = c(c-1)$, $g(c) = 1$, $p = 1$

$y = A_0 x^c + A_1 x^{c+p} + A_2 x^{c+2p}$

$\rightarrow k = 0$, $A_1 = (-1)^1 A_0 \dfrac{1}{(c+1)(c+1-1)}$

$$= \frac{-A_0}{c(c+1)}$$

note::

$f(c) = c(c-1)$

$f(c+p) = (c+p)(c+p-1)$

$f(c+2p) = (c+2p)(c+2p-1)$

$K=1$, $A_2 = (-1)^2 A_0 \dfrac{1}{[c(c+1)][(c+2)(c+1)]} = \dfrac{A_0}{c(c+1)^2(c+2)} =$ عوضي

$$y = A_0 x^c + A_1 x^{c+1} + A_2 x^{c+2} + \cdots$$

$$= A_0 x^c - \frac{A_0}{c(c+1)} x^{c+1} + \frac{A_0}{c(c+1)^2(c+2)} x^{c+2} + \cdots$$

. $f(c) = c(c-1) = 0 \rightarrow c_1 = 0$ (smallest), $c_2 = 1$ (largest)

$$\rightarrow y = A_0 x^c \left[1 - \frac{x}{c(c+1)} + \frac{x^2}{c(c+1)^2(c+2)} + \cdots \right]$$

$$y_1 = A_0 x \left[1 - \frac{x}{1(2)} + \frac{x^2}{(1)(2)^2(3)} \right]$$

(use largest)

$$= A_0 x \left[1 - \frac{x}{2} + \frac{x^2}{12} \right]$$

$$y_2 = A_0 x^c - \frac{A_0 x^{c+1}}{c(c+1)} + \cdots \Bigg|_{c=\text{largest}} + (c - c_1) \left[\frac{A_0 x^{c+1}}{c(c+1)^2} + \frac{A_0 x^{c+1}}{c^2(c+1)} \right]$$

Laplace Transformation

Change t-domain $\longrightarrow$ s-domain

$$f(t) \longrightarrow F(s)$$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

example (1)

$f(t) = 1$

$$L[1] = \int_0^\infty e^{-st} (1)\, d(t) = \frac{-1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s}$$

example (2)

$$L[k] = \int_0^\infty e^{-st} k\, dt = k \int_0^\infty e^{-st} dt = \frac{k}{s}$$

example (3)

$$L[e^{\alpha t}] = \int_0^\infty e^{-st} e^{\alpha t}\, dt = \int_0^\infty e^{(\alpha - s)t}\, dt = \frac{1}{\alpha - s} e^{-(s-\alpha)t} \Big|_0^\infty = \frac{1}{(s - \alpha)}$$

example (4)

$f(t) = kt \longrightarrow F(s) = ?$

$$L[kt] \int_0^\infty e^{-st} k\,t\,dt = k\int_0^\infty t\,e^{-st}\,dt$$

$u = t \qquad dv = e^{-st}\,dt$

$du = dt \qquad v = -\frac{1}{s} e^{-st}$

$$\int u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du = \frac{-t}{s} e^{-st}\Big|_0^\infty + \int_0^\infty \frac{1}{s} e^{-st}\,dt$$

$$= -\frac{1}{s^2} e^{-st}\Big|_0^\infty = \frac{1}{s^2}$$

$$L[kt] = \frac{k}{s^2}$$

example (5)

$$f(t) = \sin \omega t \longrightarrow F(s) = \frac{\omega}{s^2+\omega^2} \quad \#$$

متطابقة $\sin \omega t = \dfrac{e^{i\omega t} - e^{-i\omega t}}{2i}$

$$L[f(t)] = L[\sin \omega t] = L\left[\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right] = \frac{1}{2i} L[e^{i\omega t} - e^{-i\omega t}]$$

$$L[\sin \omega t] = \frac{1}{2i}\int_0^\infty e^{-st}[e^{i\omega t} - e^{-i\omega t}]\,dt$$

$$= \frac{1}{2i}\left\{\int_0^\infty e^{-st} e^{i\omega t}\,dt - \int_0^\infty e^{-st} e^{-i\omega t}\,dt\right\}$$

$$= \frac{1}{2i}\left\{\int_0^\infty e^{-(s-i\omega)t}\,dt - \int_0^\infty e^{-(s+i\omega)t}\,dt\right\}$$

$$= \frac{1}{2i}\left\{\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right\}$$

$$= \frac{1}{2i}\left\{\frac{(s+i\omega)-(s-i\omega)}{s^2+\omega^2}\right\}$$

$$= \frac{1}{2i}\left\{\frac{2i\omega}{s^2+\omega^2}\right\} = \frac{\omega}{s^2+\omega^2}$$

$$(s-i\omega)(s+i\omega) = s^2 - i^2\omega^2 = s^2 - (\sqrt{-1})^2\omega^2$$

$$= s^2+\omega^2$$

[HOMEWORK]

$$F(t) = te^{-\infty t}$$

$$F[te^{-\infty t}] = \int_0^\infty e^{-st}(te^{-\infty t})\,dt$$

$$= \int_0^\infty te^{-(s+\infty)t}\,dt$$

$u = t \qquad dv = e^{-(s+\infty)t}$

$du = dt \qquad v = \dfrac{e^{-(s+\infty)t}}{-(s+\infty)}$

$$\int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$= \frac{-te^{-(s+\infty)t}}{s+\infty}\Bigg|_0^\infty + \int_0^\infty \frac{e^{-(s+\infty)t}}{(s+\infty)}\,dt$$

$$= 0 + \frac{-e^{-(s+\infty)t}}{(s+\infty)^2}\Bigg|_0^\infty$$

$$= \frac{1}{(s+\infty)^2}$$

$$L\left[\frac{dx}{dt}\right] = \int_0^\infty \left(\frac{dx}{dt}\right) e^{-st}\, dt$$

let :

$u = e^{-st}$ $\qquad du = -s\, e^{-st}\, dt$

$dv = \frac{dx}{dt}$ $\qquad v = x$

$$\int_0^\infty \left(\frac{dx}{dt}\right) e^{-st}\, dt = x e^{-st}\Big|_0^\infty - \int_0^\infty -s x\, e^{-st}\, dt$$

$$= -x(t=0) + s\int_0^\infty x^{(t)} e^{-st}\, dt$$

note :

$$L[f(t)] = \int_0^\infty f(t) e^{-st}\, dt = F(s)$$

$$= -x(t=0) + s x(s)$$

$$\boxed{L\left[\frac{dx}{dt}\right] = s x(s) - x(t=0)}$$

$$L\left[\frac{d^2x}{dt^2}\right] = \int_0^\infty \left(\frac{d^2x}{dt^2}\right) e^{-st}\, dt$$

$u = e^{-st}$ $\qquad dv = \frac{d^2x}{dt^2}\, dt$

$du = -s e^{-st} dt$ $\qquad v = \frac{dx}{dt} = x'(t)$

$$L\left[\frac{d^2x}{dt^2}\right] = x'(t) e^{-st}\Big|_0^\infty - \int_0^\infty -s x'(t) e^{-st}\, dt$$

$$= -x'(t=0) + s\int_0^\infty x'(t) e^{-st}\, dt$$

$$= -x'(t=0) + s\left[s x(s) - x(t=0)\right]$$

$$\boxed{L\left[\frac{d^2x}{dt^2}\right] = s^2 x(s) - s x(0) - x'(0)}$$

Solve : $\frac{d^2y}{dx^2} + 4y = 3$ $\quad$ * given that $y(0) = y'(0) = 1$

$$\left[s^2 y(s) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{1}\right] + 4y(s) = \frac{3}{s}$$

$$s^2 y(s) + 4y(s) - s - 1 = 3/s$$

$$y(s)\left[s^2 + 4\right] = 3/s + s + 1$$

$$y(s) = \frac{3/s + s + 1}{s^2 + 4} = \frac{3/s}{s^2+4} + \frac{s}{s^2+4} + \frac{1}{s^2+4}$$

$$= \frac{3}{s[s^2+4]} + \frac{s}{s^2+4} + \frac{1}{s^2+4}$$

label (online)

$$y(t) = (\ \ ?\ \ ) + (\cos 2t) + \frac{1}{2}\sin 2t$$

$F(t)$	$F(s)$
$\frac{1}{\alpha}\sin\alpha t$	$\frac{1}{\alpha}\frac{\alpha}{s^2+\alpha^2}$
	$\frac{1}{s(s^2+\alpha^2)}$

Solve : $dy/dt + y = 0 \rightsquigarrow dy/-y = dt = -\ln y = t\ ,\ y = e^{-t}$

$y'(0) = 1$

$sy(s) - \cancel{y(0)} + y(s) = 0$ 1

$y(s)[s+1] = 1$

$y(s) = \frac{1}{s+1} \rightsquigarrow y(t) = e^{-t}$

$$F(s) = \frac{Z(s)}{R(s)} = \frac{Z(s)}{(s-P_1)^n(s-P_1)^{n-1}\cdots(s-P_1)(s-P_2)^m(s-P_2)^{m-1}\cdots}$$

$$= \frac{A_1}{(s-P_1)^n} + \frac{A_2}{(s-P_1)^{n-1}} + \frac{A_3}{(s-P_1)^{n-2}} \cdots + \frac{B_1}{(s-P_2)^m} + \frac{B_2}{(s-P_2)^{m-1}} + \cdots + \frac{C}{\cdots}$$

$$A_m = \lim_{s\to P_1}\left[\frac{d^{m-1}}{ds^{m-1}}(s-P_1)^n F(s)\right]\frac{1}{(m-1)!}$$

---

example :

$$\frac{3}{(x-2)(x+4)} = \frac{A}{(x-2)} + \frac{B}{(x+4)}$$

$$= \frac{A(x+4) + B(x-2)}{(x-2)(x+4)}$$

$$= \frac{Ax + 4A + Bx - 2B}{(x-2)(x+4)}$$

$(A+B)x + (4A-2B) = 3 + 0x$

$A+B = 0 \rightsquigarrow A = -B$

$4A - 2B = 3 \rightsquigarrow 4(-B) - 2B = 3$

$-4B - 2B = 3 \rightsquigarrow B = \frac{-1}{2}\ ,\ A = \frac{1}{2}$

$$\frac{3}{(x-2)(x+4)} = \frac{1/2}{(x-2)} + \frac{-1/2}{(x+4)}$$

---

example :

$$F(s) = \frac{1}{(s-a)(s-b)} = \frac{A_1}{s-a} + \frac{B_1}{(s-b)}$$

$$A_1 = \lim_{s \to a} (s-a) \left[\frac{1}{(s-a)(s-b)} \right] = \frac{1}{a-b}$$

$$B_1 = \lim_{s \to b} \left[(s-b) \frac{1}{(s-a)(s-b)} \right] = \frac{1}{b-a}$$

$$F(s) = \frac{1/(a-b)}{s-a} + \frac{1/(b-a)}{s-b}$$

$b = -4$

$a = 2$

---

example :

$$F(s) = \frac{1}{(s-a)^2(s-b)} = \frac{A_1}{(s-a)^2} + \frac{A_2}{(s-a)} + \frac{B_1}{(s-b)}$$

$$A_1 = \lim_{s \to a} \left[(s-a)^2 \frac{1}{(s-a)^2(s-b)} \right] = \frac{1}{(a-b)}$$

$$A_2 = \lim_{s \to a} \left[\frac{d}{ds} (s-a)^2 \frac{1}{(s-a)^2(s-b)} \right] = \lim_{s \to a} \left[\frac{d}{ds} \left(\frac{1}{s-b} \right) \right]$$

$$= \left. \frac{-(1)(1)}{(s-b)^2} \right|_{s \to a} = \frac{-1}{(a-b)^2}$$

$$B_1 = \lim_{s \to a} \left[(s-b) \frac{1}{(s-a)^2(s-b)} \right] = \frac{1}{(b-a)^2}$$

$$F(s) = \frac{1/a-b}{(s-a)^2} + \frac{-1/(a-b)^2}{(s-a)} + \frac{1/(b-a)^2}{s-b}$$

$$F(t) = \left(\frac{1}{a-b}\right) t e^{at} - \frac{1}{(a-b)^2} e^{at} + \frac{1}{(b-a)^2} e^{bt}$$

$F(t)$	$F(s)$
k	k/s
t	$1/s^2$
$\frac{t^{n-1}}{(n-1)!}$	$1/s^n$, $n=1,2,\dots$
$\frac{1}{\sqrt{\pi t}}$	$1/\sqrt{s}$
$2\sqrt{\frac{t}{\pi}}$	$s^{-3/2}$
$\frac{1}{\alpha}\sin\alpha t$	$1/s^2+\alpha^2$
$\cos\alpha t$	$s/s^2+\alpha^2$
$\frac{1}{\alpha}\sinh(\alpha t)$	$1/s^2-\alpha^2$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt = \lim_{b\to\infty}\int_0^b e^{-st} f(t)\,dt$$

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	e^{at}	$\frac{1}{s-a}$
3.	t^n, $n=1,2,3,\dots$	$\frac{n!}{s^{n+1}}$	4.	t^p, $p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}}$, $n=1,2,3,\dots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$e^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$e^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n e^{at}$, $n=1,2,3,\dots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{e^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	e^{-cs}
27.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$	28.	$u_c(t)g(t)$	$e^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$e^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t)$, $n=1,2,3,\dots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

Frequency shift property

Scaling Property

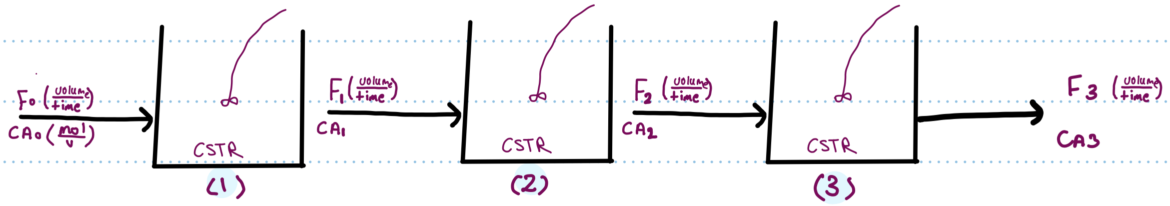
Time-shift property

Frequency differentiation property

Time-integration property

Time-periodic

Time-differentiation property



$A \xrightarrow{R} B \quad , \quad r = RC_A$

$F_0 = F_1 = F_2 = F_3$ | $C_{A1}(0) = C_{A2}(0) = C_{A3}(0) = 0$

$V_1 = V_2 = V_3$

Reactor (1) :

mol in $-$ mol out $+$ cons. $=$ acc

$C_A = \frac{N_A}{V} \longrightarrow N_A = VC_A$

$$\frac{N_{A0}}{time} - \frac{N_A}{time} - r_A V = \frac{dN_A}{dt}$$

$$F(C_{A0} - C_A) - RC_A V = V\frac{dC_A}{dt}$$

$$\frac{Volume}{time}\left(\frac{mol}{Volume} - \frac{mol}{Volume}\right)$$

$$V\frac{dC_A}{dt} = FC_{A0} - [FC_{A1} + RC_A V]$$

$$= FC_{A0} - [F + RV]C_{A1}$$

$$\boxed{\frac{dC_{A1}}{dt} + \frac{[F+RV]}{V} C_{A1} = \frac{F\,C_{A0}}{V}}$$

$$\frac{dy}{dx} + \alpha_1 y = \beta_1 (1)$$

$$F(C_{A1} - C_{A2}) - RC_{A2}V$$

$$\boxed{\frac{dC_{A2}}{dt} + \frac{[F+RV]}{V} C_{A2} = \frac{F C_{A1}}{V}}$$

$$\frac{dz}{dx} + \alpha_2 z = \beta_2 y$$

$$\boxed{\frac{dC_{A3}}{dt} + \frac{[F+RV]}{V} C_{A3} = \frac{F}{V} C_{A2}}$$

$$\frac{dw}{dt} + \alpha_3 w = \beta_3 z$$

$$dy/dt + \alpha_1 y = \beta_1$$
$$dz/dt + \alpha_2 z = \beta_2 y$$
$$dw/dt + \alpha_3 w = \beta_3 z$$

$$\boxed{\frac{dC_{A_1}}{dt} + \frac{[F+RV]}{V} C_{A_1} = \frac{F\,C_{A_0}}{V}}$$

Laplace Transformation :

$$\frac{dC_{A_1}}{dt} + \alpha_1 C_{A_1} = \beta_1$$

$$[S C_{A_1}(S) - C_{A_1}(0)] + \alpha_1 C_{A_1}(S) = \frac{\beta_1}{S}$$

0

$$S C_{A_1}(S) + \alpha_1 C_{A_1}(S) = \frac{\beta_1}{S}$$

$$C_{A_1}[S + \alpha_1] = \frac{\beta_1}{S}$$

$$C_{A_1} = \frac{\beta_1}{S(S+\alpha_1)} = \frac{A_1}{S} + \frac{B_1}{S+\alpha_1}$$

$$A_1(S+\alpha_1) + B_1(S) = \beta_1 + 0S$$

$$A_1\alpha_1 + A_1 S + B_1(S) = \beta_1 + 0S$$

$$\left.\begin{array}{l} A_1\alpha_1 = \beta_1 \\ (A_1+B_1)S = 0S \end{array}\right] \quad \begin{array}{l} A_1 = \beta_1/\alpha_1 \\ B_1 = -A_1 = -\beta_1/\alpha_1 \end{array}$$

$$C_{A_1} = \frac{\beta_1/\alpha_1}{S} + \frac{-\beta_1/\alpha_1}{S+\alpha_1}$$

$$C_{A_1}(S) = \frac{\beta_1}{\alpha_1}\left[\frac{1}{S} - \frac{1}{S+\alpha_1}\right]$$

$$C_{A_1}(t) = \frac{\beta_1}{\alpha_1}\left[1 - e^{-\alpha_1 t}\right] \rightsquigarrow \text{table}$$

1) $dC_{A1}/dt + \alpha_1 C_{A1} = \beta_1$

2) $dC_{A2}/dt + \alpha_2 C_{A2} = \beta_2 C_{A1}$

3) $dC_{A3}/dt + \alpha_3 C_{A3} = \beta_3 C_{A2}$

1) $C_{A1}(s) = \frac{\beta_1}{s(s+\alpha_1)} = \frac{\beta_1}{\alpha_1}\left[\frac{1}{s} - \frac{1}{s+\alpha_1}\right] \rightarrow C_{A1}(t) = \frac{\beta_1}{\alpha_1}\left[1 - e^{-\alpha_1 t}\right]$

2) $[sC_{A2}(s) - \cancel{C_{A2}(0)}^{\,0}] + \alpha_2 C_{A2}(s) = \frac{\beta_2\beta_1}{s(s+\alpha_1)} \rightarrow (s+\alpha_2)C_{A2}(s) = \frac{\beta_1\beta_2}{s(s+\alpha_1)} \rightarrow$

$\rightarrow C_{A2}(s) = \frac{\beta_1\beta_2}{s(s+\alpha_1)(s+\alpha_2)} \rightsquigarrow C_{A2}(s) = \frac{A_1}{s} + \frac{B_1}{s+\alpha_1} + \frac{C_1}{s+\alpha_2} = \frac{\beta_1\beta_2}{s(s+\alpha_1)(s+\alpha_2)}$

$A_1 = \lim_{s\to 0} \cancel{s}\,\frac{\beta_1\beta_2}{\cancel{s}(s+\alpha_1)(s+\alpha_2)} = \frac{\beta_1\beta_2}{\alpha_1\alpha_2}$

$B_1 = \lim_{s\to 0} \cancel{(s--\alpha_1)}\,\frac{\beta_1\beta_2}{s\cancel{(s+\alpha_1)}(s+\alpha_2)} = \frac{-\beta_1\beta_2}{(\alpha_2-\alpha_1)\alpha_1}$

$C_1 = \lim_{s\to -\alpha_2} \cancel{(s+\alpha_2)}\,\frac{\beta_1\beta_2}{s(s+\alpha_1)\cancel{(s+\alpha_2)}} = \frac{-\beta_1\beta_2}{(\alpha_1-\alpha_2)\alpha_2}$

$C_{A2}(s) = \beta_1\beta_2\left[\frac{1/\alpha_1\alpha_2}{s} - \frac{1}{\alpha_1(\alpha_2-\alpha_1)}\frac{1}{s+\alpha_1} - \frac{1}{(\alpha_1-\alpha_2)\alpha_2}\frac{1}{s+\alpha_2}\right]$

$C_{A2}(t) = \beta_1\beta_2\left[\frac{1}{\alpha_1\alpha_2} - \frac{1}{\alpha_1(\alpha_2-\alpha_1)}e^{-\alpha_1 t} - \frac{1}{(\alpha_1-\alpha_2)\alpha_2}e^{-\alpha_2 t}\right]$

3) $[sC_{A3}(s) - \cancel{C_{A3}(0)}^{\,0}] + \alpha_3 C_{A3}(s) = \frac{\beta_3\beta_2\beta_1}{s(s+\alpha_1)(s+\alpha_2)}$

$C_{A3}(s)[s+\alpha_3] = \ldots$

$C_{A3}(s) = \frac{\beta_1\beta_2\beta_3}{s(s+\alpha_1)(s+\alpha_2)(s+\alpha_3)} = \frac{A_1}{s} + \frac{B_1}{s+\alpha_1} + \frac{C_1}{s+\alpha_2} + \frac{D_1}{s+\alpha_3}$

$A_1 = \frac{\beta_1\beta_2\beta_3}{\alpha_1\alpha_2\alpha_3}, \quad B_1 = \frac{-\beta_1\beta_2\beta_3}{\alpha_1(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)}, \quad C_1 = \frac{-\beta_1\beta_2\beta_3}{\alpha_2(\alpha_3-\alpha_2)(\alpha_1-\alpha_2)}$

$D_1 = \frac{-\beta_1\beta_2\beta_3}{\alpha_3(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)}$

$C_{A3}(t) = A_1 + B_1 e^{-\alpha_1 t} + C_1 e^{-\alpha_2 t} + D_1 e^{-\alpha_3 t}$

$= \frac{\beta_1\beta_2\beta_3}{\alpha_1\alpha_2\alpha_3}\left[1 - \frac{\alpha_2\alpha_3}{(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)}e^{-\alpha_1 t} - \frac{\alpha_1\alpha_3}{(\alpha_3-\alpha_2)(\alpha_1-\alpha_2)}e^{-\alpha_2 t} - \frac{\alpha_1\alpha_2}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)}e^{-\alpha_2 t}\right]$

example:

$dy_1/dt - 2y_1 = 1 \rightarrow y_1(0) = 0,\ y_2(0) = 1$

$dy_2/dt + y_2 = y_1$

$s y_1(s) - \cancel{y_1(0)}^{0} - 2y_1(s) = \frac{1}{s}$

$y_1(s) = \frac{1}{s(s-2)} = \frac{A_1}{s} + \frac{B_1}{s-2}$

$y_1(t) = -\frac{1}{2}[1 - e^{2t}]$

$[s y_2(s) - \cancel{y_2(0)}^{1}] + y_2(s) = y_1(s)$

$[s+1]\, y_2(s) = \frac{1}{s(s-2)} + \frac{1}{1}$

$= \frac{1 + s(s-2)}{s(s-2)}$

$= \frac{s^2 - 2s + 1}{s(s-2)}$

$y_2(s) = \frac{s^2 - 2s + 1}{s(s+1)(s-2)} = \frac{A_1}{s} + \frac{B_1}{s+1} + \frac{C_1}{s-2}$

$A_1 = \lim_{s \to 0} s \frac{s^2 - 2s + 1}{s(s+1)(s-2)} = -\frac{1}{2}$

$C_1 = \lim_{s \to 2} (s-2) \frac{s^2 - 2s + 1}{s(s+1)(s-2)} = \frac{1}{6}$

$B_1 = \lim_{s \to -1} (s+1) \frac{s^2 - 2s + 1}{s(s+1)(s-2)} = \frac{4}{3}$

$A_1 = -\frac{1}{2}, \quad B_1 = \frac{1}{2}$

$y_2(s) = -\frac{1}{2}\left(\frac{1}{s}\right) + \left(\frac{4}{3}\right)\frac{1}{s+1} + \frac{1}{6}\left(\frac{1}{s-2}\right)$

$y_2(t) = -\frac{1}{2} + \frac{4}{3}e^{-t} + \frac{1}{6}e^{2t}$

example:

$dy_1/dt + 4y_1 = y_2$

$dy_2/dt + 10y_2 = y_1$

$\rightsquigarrow y_1(0) = y_2(0) = 0$

$[sy_1(s) + \cancel{y_1(0)}^{0}] + 4y_1(s) = y_2(s)$

$[sy_2(s) + \cancel{y_2(0)}^{0}] + 10y_2(s) = y_1(s)$

$y_1[s+4] = y_2$

$y_2[s+10] = y_1$

$y_1[s+4][s+10] = y_1$

$y_1[\quad][\quad] - y_1 = 0$

$y_1[(s+4)(s+10) - 1] = 0$

$y_1(s) = 0 \rightarrow y_1(t) = 0$ } trivial

$y_2[s+10] = 0 \rightarrow y_2(t) = 0$ } solution

$\rightsquigarrow y_1(0) = 1, \; y_2(0) = 0$

$$y_2\left[\frac{(s+10)(s+4)-1}{s+4}\right] = \frac{1}{s+4}$$

$$y_2 = \frac{(s+4)^2}{(s+10)(s+4)-1} = \frac{1}{(s+3.8)(s+10.15)} = \frac{A}{s+3.8} + \frac{B}{s+10.15} = \frac{1/6.35}{s+3.8} + \frac{-1/6.35}{s+10.15}$$

$A(s+10.15) + B(s+3.8) = 1$

$(A+B)s + 10.15A + 3.8B = 1 + 0s$

$A = -B$

$10.15A + 3.8B = 1$

$-10.15B + 3.8B = 1$

$-6.35B = 1 \rightsquigarrow B = \frac{-1}{6.35}$

$A = \frac{1}{6.35}$

$(s+10)(s+4) - 1 = s^2 + 14s + 39$

$$s = \frac{-14 \mp \sqrt{196 - 4(1)(39)}}{2}$$

$$= \frac{-14 \mp 6.3}{2}$$

$s_1 = -3.8, \; s_2 = -10.15$

$$y_2(s) = \frac{1}{6.35}\left[\frac{1}{s+3.8} - \frac{1}{s+10.15}\right]$$

$$y_2(t) = \frac{1}{6.35}\left[e^{-3.8t} - e^{-10.15t}\right]$$

$sy_1(s) - y_1(0) + 4y_1(s) = y_2(s)$

$$y_1(s)[s+4] - 1 = \frac{1}{(s+3.6)(s+10.15)}$$

$$y_1(s)[s+4] = \frac{(s+10)(s+4)}{(s+3.8)(s+10.15)}$$

$$y_1(s) = \frac{s+10}{(s+3.8)(s+10.15)}$$

$$= \frac{A}{s+3.8} + \frac{B}{s+10.15}$$

example :-

$$\frac{d^3y}{dt^3} \qquad L\left[\frac{d^3y}{dt^3}\right] = \int_0^\infty \frac{d^3y}{dt^3}$$

$u = e^{-st}$ $\qquad du = -se^{-st}$

$dv = \frac{d^3y}{dt^3}$ $\qquad dt \quad v = \frac{d^2y}{dt^2}$

$$e^{-st}\frac{d^2y(t)}{dt^2}\Big|_0^\infty - \int\left[\frac{d^2y}{dt^2}\right](-se^{-st})$$

$$y''(0) + s\int\left(\frac{d^2y}{dt^2}\right)e^{-st}$$

$$y''(0) + s[s^2y(s) - sy(0) - y'(0)]$$

$$s^3y(s) - s^2y(0) - sy'(0) + y''(0)$$

$L[y'] = sy(0) - y(0)$

$L[y''] = s^2y(s) - sy(0) - y'(0)$

$L[y'''] = s^3y(s) - s^2y(0) - sy'(0) - y''(0)$

$L[y''''] = s^4y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)$

$L[y'''''] = s^5y(s) - s^4y(0) - s^3y'(0) - s^2y''(0) - sy'''(0) - y^{IV}(0)$

$\frac{d^3y}{dt^3} + y = 1 \qquad y(0) = y'(0) = y''(0) = 1$

$$s^3y(s) - s^2 - s - 1 + y(s) = \frac{1}{s}$$

$$y(s)[s^3+1] = \frac{1}{s} + \frac{s^2}{1} + \frac{s}{1} + \frac{1}{1}$$

$$= \frac{1 + s^3 + s^2 + s}{s}$$

$$y(s)[s^3+1] = \frac{s^3+s^2+s+1}{s}$$

$$y(s) = \frac{s^3+s^2+s+1}{s(s^3+1)}$$ $\rightsquigarrow$ 4 terms will give me continue the solution

Integral functions:

1) error functions

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\beta^2} d\beta$$

$$erf(x) = 1 - (\alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3) e^{-x^2}$$

$$t = \frac{1}{1 + px}$$

$p = 0.47047$

$\alpha_1 = 0.34802$

$\alpha_2 = -0.09587$

$\alpha_3 = 0.74785$

example:

$$\int_0^5 e^{-x^2} dx \leftrightarrow erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx \qquad t = \frac{1}{1 + .47(5)} = 0.298$$

$$\int_0^5 e^{-x^2} dx = \frac{\sqrt{\pi}}{2} erf(x)$$

$$= \frac{\sqrt{\pi}}{2} \left[1 - \{ 0.348(0.298) - 0.095(0.298)^2 + 0.747(.298)^3 e^{-5^2} \} \right]$$

$$= \frac{\sqrt{\pi}}{2} \left[1 - \{ 0.103 - 8.4 * 10^{-3} + 0.0198 \} (1.38 * 10^{-11}) \right]$$

$$= \frac{\sqrt{\pi}}{2} \left[1 - 1.58 * 10^{-12} \right] = \frac{\sqrt{\pi}}{2}$$

$erf(0) = ?$ Home work

$$erf(\infty) = 1$$

$$erfi(x) = \frac{2}{\sqrt{\pi}} \int e^{-x^2} dx$$

$$\frac{d}{dx} erf(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

$$* \int erf(x) = x\, erf(x) - \frac{1}{\sqrt{\pi}} e^{-x^2}$$

$$erf(x) + erfc(x) = 1 \rightsquigarrow erfc(x) = 1 - erf(x)$$

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$erf(0) = \frac{2}{\sqrt{\pi}} \int_0^0 e^{-t^2} dt = 0 \rightsquigarrow \boxed{erf(0) = 0}$$

Integral functions

2) Gamma functions.

$$\Gamma(x) = \frac{1}{x}\prod_{n=1}^{\infty}\frac{(1+1/n)^x}{(1+x/n)} = \frac{1}{x}\left[\left\{\frac{(1+1/1)^x}{(1+x/1)}\right]\left[\frac{(1+1/2)^x}{(1+x/2)}\right\}\left\{\frac{(1+1/3)^x}{(1+x/3)}\right\}\cdots\right.$$

$$= \int_0^x t^{x-1}e^{-t}\,dt$$

$$\Gamma(x+1) = x\Gamma(x)$$

for $x =$ integer

$$\Gamma(n) = (n-1)\,\Gamma(n-1)$$
$$= (n-1)(n-2)\,\Gamma(n-2)$$
$$= (n-1)(n-2)(n-3)\,\Gamma(n-3)\cdots\Gamma(1)$$
$$= (n-1)!$$

example:

$$\Gamma(2) = (2-1)! = 1$$
$$\Gamma(5) = 4\times3\times2\times1$$

Integral functions

3) Beta function

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

$$= \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

example:

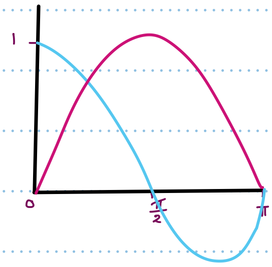
$$\beta(3,7) = \frac{\Gamma(3)\,\Gamma(7)}{\Gamma(3+7)} = \frac{(2\times1)(\cancel{6\times5\times4\times3}\times1)}{9\times8\times7\times\cancel{6\times5\times4\times3\times2\times1}} = \frac{1}{9\times4\times7}$$

Fourier Series :

any function , $f(x)$ can be written in the form of Fourier series

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad -\pi \leq x \leq \pi$$

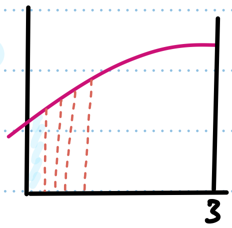
$$f(x) = \frac{1}{2} a_0 + [a_1 \cos x + b_1 \sin x] + [a_2 \cos 2x + b_2 \sin 2x] + \cdots$$



multiplay by $\cos mx$:

$$f(x) = \frac{1}{2} a_0 \cos mx + \sum_{1}^{\infty} a_n \cos nx \cos mx + \sum_{1}^{\infty} b_n \sin nx \cos mx$$

مجموع الخطوط



note : (take it as it is)

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \qquad m \neq n$$

$$\int_{-\pi}^{\pi} \cos nx \cos nx \, dx = \int_{-\pi}^{\pi} \cos^2 nx \, dx \qquad \text{when } m = n$$

$$\int_{-\pi}^{\pi} \cos nx \sin mx = 0 \qquad \text{any value of } n \,\&\, m$$

$$\int (\cos mx) f(x) = \frac{1}{2} a_0 \cos mx + \int_1^{\infty} a_n \cos nx \cos mx \, dx + \int_1^{\infty} b_n \sin nx \cos mx \, dx$$

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \int_{-\pi}^{\pi} a_0 \cos mx + a_n \int_{-\pi}^{\pi} \cos nx \cos mx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx \, dx$$

For $m \neq n \Rightarrow \int_{-\pi}^{\pi} f(x) \cos mx \, dx = \frac{1}{2} a_0 \int_{-\pi}^{\pi} \cancel{\cos mx \, dx}^{\,0} \longrightarrow$ Trivial Solution

for $m = n \Rightarrow \int_{-\pi}^{\pi} f(x) \cos mx \, dx = \frac{1}{2} a_0 \int_{-\pi}^{\pi} \cancel{\cos mx \, dx}^{\,0} + a_n \int_{-\pi}^{\pi} \cancel{\cos^2 mx \, dx}^{\,\pi}$

Should be cosnx?

$$\boxed{a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos \overset{n}{m} x \, dx}$$

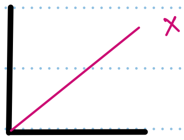
$$\boxed{b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx}$$

$$\boxed{a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx}$$

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad -\pi \leqslant x \leqslant \pi$$

example:-

$$f(x) = \begin{cases} 0 & -\pi \leqslant x < 0 \\ x & 0 \leqslant x \leqslant \pi \end{cases}$$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} 0 \cos mx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x \cos mx\,dx = \frac{1}{\pi n^2}\Big[\cos n\pi - 1\Big]_0^{\pi}$$ → (تم التعويض)

$$b_n = \frac{1}{\pi}\int_{\pi}^{0} 0 \sin nx\,dx + \int_{0}^{\pi} x \sin nx\,dx = \frac{-\cos nx}{n}\Bigg|_0^{\pi}$$

note:-

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad -\pi \leqslant x \leqslant \pi$$

$$= \frac{1}{2} a_0 + a_1 \cos x + b_1 \sin x$$
$$+ a_2 \cos 2x + b_2 \sin 2x$$
$$+ a_3 \cos 3x + b_3 \sin 3x$$
$$+ \cdots$$

Continue the solution of the example:-

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{\pi (1)^2}\Big[\cos x - 1\Big]_0^{\pi} \cos x + \left[\frac{-\cos x}{1}\right]_0^{\pi} \sin x$$

$$+ \frac{1}{\pi (2)^2}\Big[\cos 2x - 1\Big]_0^{\pi} \cos x + \left[\frac{-\cos 2x}{2}\right]_0^{\pi} \sin x$$

$$+ \cdots$$

$$= \frac{\pi}{4} + \frac{-2}{\pi} \cos x + 2 \sin x$$
$$+ \cdots$$

Continue the solution...

example: (homework)

$f(x) = x^2$ (not correct)

$f(x)$ is an even function

since $f(-x) = (-x)^2 = x^2$

$\therefore b_n = 0$

$\rightsquigarrow$ we only need to calculate a_0 and a_n

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{\pi}\int_{0}^{\pi} x^2\,dx$$

$$a_0 = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\cdot\frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\left[\frac{x^2\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left(\frac{2\pi(-1)^n}{n^2}\right) = \frac{4(-1)^n}{n^2}$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$$

$$x^2 = \frac{\pi^2}{3} - 4\left(\cos(x) - \frac{\cos(2x)}{4} + \frac{\cos(3x)}{9} - \dots\right)$$

if we put $x = \pi$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

note:

* الحل غير معتمد

ODD Function $\rightsquigarrow$ x, x^3, $\sin x$, $\tan x$

$f(-x) = -f(x)$

$f(x) = x \quad , \quad f(-x) = -x$

even function $\rightsquigarrow$ x^2, x^4, $\cos x$

$f(-x) = f(x)$

$f(x) = x^2 \rightarrow (-1)^2 = (1)^2 = 1$

* ODD * ODD = even

$x * x^3 = x^4$

* ODD * even = ODD

$x * x^2 = x^3$

* even * even = even

$x^2 * x^4 = x^6$

$$\int_{-L}^{L} f(x)\,dx = \begin{cases} 0 & , f(x) = \text{ODD} \\ 2\int_0^L f(x)\,dx & , f(x) = \text{even} \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

---

example :

$f(x) = x + x^2$

Find Fourier series for this function

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}[x + x^2]\,dx$$

$$= \frac{1}{\pi}\left[\cancel{\int_{-\pi}^{\pi} x\,dx}^{\,0} + \int_{-\pi}^{\pi} x^2\,dx\right] = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{3}\right] = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx\,dx + \int_{-\pi}^{\pi} x^2\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{4(-1)^n}{n^2}$$

note :

$\cos nx \longrightarrow \quad n=0 \rightarrow \cos 0 = 1$

$n=1 \rightarrow \cos\pi = -1$

$n=2 \rightarrow \cos 2\pi = 1$

$n=3 \rightarrow \cos 3\pi = -1$

$\cos n\pi = (-1)^n$

$$\int_{-\pi}^{\pi} x^2\cos nx\,dx = 2\int_0^{\pi}$$

$u = x^2 \qquad dv = \cos nx\,dx$

$du = 2x\,dx \qquad v = \frac{1}{n}\sin nx$

$$\frac{x^2}{n}\sin nx\Big|_0^{\pi} - \int_0^{\pi}\frac{2}{n}x\sin nx\,dx$$

$\int_0^{\pi} x\sin nx\,dx \qquad u = x \qquad dv = \sin nx\,dx$

$du = dx \qquad v = \frac{\cos nx}{n}$

$$= -\frac{x}{n}\cos(nx)\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}\,dx$$

$$\int_0^{\pi} x^2\cos nx\,dx = \frac{x^2}{n}\sin x\Big|_0^{\pi} - \frac{2}{n}\left\{-\frac{x}{n}\cos(nx)\Big|_0^{\pi} + \frac{1}{n^2}\sin(nx)\Big|_0^{\pi}\right\}$$

$$= \frac{\pi^2}{n}\cancel{\sin(n\pi)}^{\,0} - \frac{2}{n}\left\{-\frac{\pi}{n}\cos n\pi\right\}$$

$$= \frac{2\pi}{n^2}\cos(n\pi)$$

example :

Convert $f(x) = e^x$

$$F(x) = \frac{a_0}{2} + \sum_{1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Solution::

$$\int e^{ax}\cos(bx) = e^{ax}\frac{[a\cos(bx)+b\sin(bx)]}{a^2+b^2}$$

$$a_0 = \frac{e^{\pi}-e^{-\pi}}{\pi}$$

$$a_n = \frac{e^x}{\pi}\left[\frac{\cos(nx)+n\sin(nx)}{1+n^2}\right]_{-\pi}^{\pi}$$

$$= e^{\pi}\left[\frac{\overbrace{\cos(n\pi)}^{(-1)^n}+n\overbrace{\sin(n\pi)}^{0}}{1+n^2}\right] - e^{-\pi}\left[\frac{\overbrace{\cos(-n\pi)}^{(-1)^n}+n\overbrace{\sin(-n\pi)}^{0}}{1+n^2}\right]$$

$$= \frac{e^{\pi}(-1)^n}{1+n^2} - \frac{e^{-\pi}(-1)^n}{1+n^2} = \frac{(-1)^n[e^{\pi}-e^{-\pi}]}{\pi(1+n^2)}$$

$$\int e^{ax}\sin(bx)\,dx = e^{ax}\frac{[a\sin(bx)-b\cos(bx)]}{a^2+b^2}$$

$$b_n = \frac{1}{\pi}\int e^x\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left\{e^x\frac{[\sin(nx)-n\cos(nx)]}{1+n^2}\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{e^{\pi}\frac{[\overbrace{\sin(n\pi)}^{0}-n\cos(n\pi)]}{1+n^2} - e^{-\pi}\frac{[\overbrace{\sin(-n\pi)}^{0}-n\cos(-n\pi)]}{1+n^2}\right.$$

$$= \frac{1}{\pi(1+n^2)}[-ne^{\pi}(-1)^n+ne^{-\pi}(-1)^n]$$

$$b_n = \frac{n(-1)^n}{\pi(1+n^2)}[e^{-\pi}-e^{\pi}]$$

متطابقات الجدول لأي امتحان

Convert $f(x) = \sin x$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin x\,dx = -\frac{1}{\pi}\cos x\Big|_{-\pi}^{\pi} = -\frac{1}{\pi}[-1--1] = 0$$

$$a_n = \frac{1}{\pi}\int \sin x\cos(nx)\,dx = 0$$

$$b_n = \frac{1}{\pi}\int \sin x\sin(nx)\,dx$$

$e^x \ln x$ → can't be solved

$\left.\begin{array}{l} e^x \ln x \\ 3^x \\ \dfrac{e^x - e^{-x}}{2} \end{array}\right] \rightarrow$ final

* جدول المتكاملات
* جدول L [ ]

يدخلوا معنا للفاينل

will be solved easily

only bn will be calculated

since it's an odd function

partial Diff equ. :

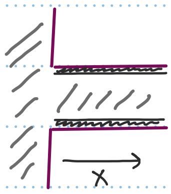
$$\frac{dy}{dx} + f(x)\,y = g(x) \rightarrow \text{ODE}$$

one independant variable

$$\frac{\partial y}{\partial t} = \alpha \frac{\partial y}{\partial x} \rightarrow \text{PDE}$$

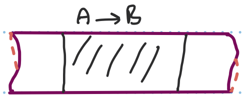
two or more independent varible

→ heat flux in (fin)



$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$\frac{\partial C_o}{\partial t} = D_{AB} \frac{\partial^2 C_A}{\partial x^2}$$



$$\frac{\partial C_A}{\partial t} = D_{AB} \frac{\partial^2 C_A}{\partial x^2} + v\frac{\partial C_A}{\partial x} + K C_A$$

$$a \frac{\partial u^2}{\partial x^2} + b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} + d \frac{\partial u}{\partial x} + e \frac{\partial u}{\partial y} + f u = g.$$

$a, b, c, d, e, f, g \rightarrow$ function of (x, y)

example :

$$3x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} + (1 + 3e^y) \frac{\partial^2 u}{\partial y^2} = \cos(x, y)$$

PDE → linear → $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = x^3$

→ non linear → $\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial y}\right)^2 = 0$, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u^2 = 0$

order of PDE

first order → $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$

2nd order → $\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} = 0$

higher order → $\frac{\partial^n u}{\partial x^n} + \cdots = \cdots$

$b^2 - 4ac = ?$

case (1) :

$b^2 - 4ac < 0$ elliptic PDE

ex :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = x \frac{\partial u}{\partial x}$$

a b c

$a = 1$, $b = 1$, $c = 1$

$b^2 - 4ac = (1)^2 - 4(1)(1) = -3 < 0$

Case (2) :

$b^2 - 4ac > 0$ hyperbolic PDE

ex :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} = x \frac{\partial u}{\partial x}$$

$a = 1$, $b = 1$, $c = -1$

$b^2 - 4ac = (1)^2 - 4(1)(-1) = +5 > 0$

Case (3) :

$b^2 - 4ac = 0 \rightarrow$ Parabolic PDE

ex :

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = x \frac{\partial u}{\partial x}$$

$a = 1$, $b = 2$, $c = 1$

$b^2 - 4ac = (2)^2 - 4(1)(1) = 0$

example :

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \rightarrow \alpha \frac{\partial^2 T}{\partial x^2} - \frac{\partial T}{\partial t} = 0$$

$a = \alpha$, $b = 0$, $c = 0$

$b^2 - 4ac = 0$

1) Separation of variables

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial y^2} \rightarrow c = f(t)\, g(y)$$

$$\frac{\partial c}{\partial t} = g(y) \frac{d f(t)}{d(t)} = g f'$$

IC, at $t=0 \rightarrow C_A = C_{A0}$

BC, at $y=0$, $C_A = C_{A0}$

at $y=L$, $C_A = C_{AL}$

$$\frac{\partial c}{\partial y} = f(t) \frac{dg}{dy}$$

$$\frac{\partial^2 c}{\partial y^2} = f(t) \frac{d^2 g}{dy^2}$$

$$g f' = f g'' \rightarrow \frac{f'}{f} = \frac{g''}{g} \rightarrow \frac{1}{f}\frac{df}{dt} = \frac{1}{g}\frac{d^2 g}{dy^2}$$

$$\frac{1}{f}\frac{df}{dt} = \frac{1}{g}\frac{d^2 g}{dy^2} = \text{constant} = \lambda^2$$

$$\frac{1}{f} \cdot \frac{df}{dt} = \lambda^2 \rightarrow \int \frac{1}{f}\, df = \int \lambda^2\, dt \rightarrow \ln f = \lambda^2 t + c \rightarrow f = \alpha e^{\lambda^2 t}$$

$T(t,x) = T(t)\,X(x)$

$= ce^{-\lambda^2 t}[A\cos\lambda x + B\sin\lambda x]$

$* \dfrac{\partial T}{\partial t} = \dfrac{\partial^2 T}{\partial x^2}$ $\begin{cases} \text{ic (initial condition)} \; T(0,x) = 3\sin 2x \\ \text{Bc}_1 \text{ (boundary condition. 1)} \; T(t,0)=0 \\ \text{Bc}_2 \; T(t,1)=0 \end{cases}$

$\rightarrow T(t,x) = [A'\cos\lambda x + B'\sin\lambda x]\, e^{-\lambda^2}$

. Bc_1 :

$0 = [A'\cos\lambda(0) + B'\sin\lambda(0)]\, e^{-\lambda^2 t}$

$A'\cos\lambda(0) + B'\sin\lambda(0)^{0} = 0$

$A'\cos\lambda(0) = 0 \rightarrow \cos(0) = 1$ then A should be zero

$\boxed{A' = 0}$

$\rightarrow T(t,x) = B'\sin\lambda x\, e^{-\lambda^2 t}$

. Bc_2 :

$0 = B'\sin\lambda(1)\, e^{-\lambda^2 t}$

$B'\sin\lambda = 0$ B' should not be zero because it becomes trivial solution

$\boxed{\sin\lambda = 0}$

$\lambda = 0, \pi, 2\pi, 3\pi = n\pi \;, n = 0, 1, 2 \ldots$

$\rightarrow T(t,x) = B'\sin(n\pi x)\, e^{-n^2\pi^2 t}$

. IC :

$3\sin 2x = B'\sin(n\pi x)e^{0}$

$3\sin 2x = B'\sin(n\pi x)$

$B' = 3 \;, n\pi = 2 \rightarrow n = \frac{2}{\pi}$

$\rightarrow T(t,x) = 3\sin(\frac{2}{\pi}\cdot\pi x)\, e^{-\frac{4}{\pi^2}\pi^2 t}$

$T(t,x) = 3\sin(2x)\, e^{-4t}$

Ex. $\dfrac{\partial T}{\partial t} = \dfrac{\partial^2 T}{\partial x^2}$ $\begin{cases} \text{ic} \; T(0,x) = 2\sin 3x + 3\cos 2x \\ \quad T(t,0) = 0 \\ \text{BC}_2 \; T(t,1) = 0 \end{cases}$

$T(t,x) = T(t)\,x(x)$

$= ce^{-\lambda^2 t}[A\cos\lambda x + B\sin\lambda x]$

$= A'\cos\lambda x + B'\sin\lambda x]\, e^{-\lambda^2 t}$

$A' = 0 \;,\; \sin\lambda = 0 \rightarrow \lambda = n\pi.$

$\rightarrow T(t,x) = B'\sin(n\pi x)e^{-n^2\pi^2 t}$

. ic :

$5\sin 3x + 3\cos 2x = B'\sin(n\pi x) + 0$

by Fourier series, convert the 1st term to new sin, cos terms then sin = sin cos = 0

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ex : $\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$

ic $T(0,x) = 3\sin 2x \rightsquigarrow t=0$

Bc₁ $T(t,0)=0$ $\quad$ $Bc_2 = T(t,1)=0$

$L[dT/dt] = sT(s,x) - T(t=0)$

$L[d^2T/dx^2] = \frac{d^2T(s,x)}{dx^2}$

$L[3\sin 2x] = 3\sin 2x / s$

$\rightarrow sT(s,x) - T(0) = \frac{d^2T(s)}{dx^2}$

$sT - \frac{3\sin 2x}{s} = \frac{d^2T}{dx^2}$

$\frac{d^2T}{dx^2} - sT = -\frac{3\sin 2x}{s}$

$T = f(x,s)$

ex:

$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$ $\quad$ $T(t=0)=0$

$sT(s) - \cancel{T(0)}^{\,0} = \frac{d^2T}{dx^2}$

$\frac{d^2T}{dx^2} - sT(s) = 0$

$T = e^{rx}$, $T' = re^{rx}$, $T'' = r^2e^{rx}$

$\rightarrow r^2e^{rx} - se^{rx} = 0$

$r^2 - s = 0$, $r = \mp\sqrt{s}$

$\rightarrow T = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x}$

$* \; L^{-1}[e^{-\sqrt{s}}] = \frac{a}{2\sqrt{\pi t^3}} \; e^{-\frac{1}{4}t}$

$L^{-1}[e^{\sqrt{s}}] = \infty$ $\quad$, $T = A(\infty) + B\,\frac{a}{2\sqrt{\pi t^3}}\,e^{-\frac{1}{4}t\,x}$